Adaptive finite elements for variational inequalities with non-smooth coefficients.

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ADAPTIVE FINITE ELEMENTS FOR VARIATIONAL INEQUALITIES WITH NON-SMOOTH COEFFICIENTS.

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Abstract. We consider an elliptic variational inequality with discontinuous coefficients arising in unilateral contact mechanics in linearized elasticity. The contact zone is an internal boundary separating sub-domains with difference elastic properties. We study some discrete formulations with mixed finite element methods and we give optimal error estimates in appropriate norms, independent of the variation of the elasticity coefficients. The focus of the article is the a posteriori analysis with residual error indicators. It is achieved both for the conforming and nonconforming discretization, in a unified framework. The residual error indicators are well suited to handle non-matching meshes and the contact conditions, and they allow us to obtain sharp and robust a posteriori estimates. An adaptive solution algorithm is proposed and few numerical experiments confirming the theory are presented.

1. Introduction

The numerical implementation of contact and impact problems in solid mechanics is usually based on finite element tools [50, 29, 39]. The choice of the finite element methods which are both easy to implement, accurate from the theoretical point of view and of low cost is crucial for these simulations. Such choice requires the use of a priori and a posteriori analysis tools to design efficient discretization strategies.

There has been a lot of progress in the numerical solution of such problems since the pioneering works in the 70's, see [27, 29, 39, 50] for an exhaustive bibliography. Working with linear (and quadratic) finite elements, various discrete formulations, depending on the modelling of the contact conditions at discrete level, were addressed in many studies [29, 14, 15, 39, 50, 25, 41, 20, 9, 5, 44, 35]. In particular, a priori error estimates and numerical algorithms for solving such variational inequalities have been extensively studied. On the contrary, the a posteriori analysis and adaptive strategies have not been sufficiently developed, particularly for problems of Signorini type. In fact, most of the existing studies are devoted to obstacle problems where various error estimators are studied. We refer the reader to [24, 1, 37, 17] and the references therein.

For frictionless unilateral contact problems, the residual based method using a penalized approach is considered in [16] (see also the references therein). The study of the error in the constitutive law is performed in [21], an error indicator based on equilibrated fluxes in [49], and a residual error estimator for the Signorini problem is considered in [32, 33].

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Most previous works dedicated to unilateral contact problems consider a contact between a deformable and a rigid body or the contact between two deformable bodies, in homogeneous linearized elasticity. Our aim in this paper is to consider contact problems for nonhomogeneous linearized elasticity that rely on the study of a variational inequality with discontinuous coefficients: the elasticity coefficients are smooth on a finite number of subdomains. These models describe, for example, the unilateral crack propagation in dissimilar media or the delamination in composite materials (see [2]).

It is our goal to prove a priori and a posteriori estimates that are independent of the size of the jumps in the coefficients across the interfaces between the subdomains. We consider both the conforming and nonconforming discretization, and we perform their analysis in a unified framework. In particular we introduce new residual error estimators that take into account the contact conditions and the non-matching grids at the interfaces of the sub-domains. The residual error indicators are easily computable and we obtain sharp estimates without any saturation assumption or any extra regularity assumption. The scaling factors of the error estimators allow to handle correctly the local ratio between adjacent coefficients. As the meshes should be aligned with the discontinuities, we build a new quasi-interpolation operator of Clément type [19]. This operator and the use of appropriate norms allow us to obtain estimates for the interpolation error that are independent of the size of the jumps of the coefficients. The a posteriori analysis presented in the article covers, as largely as possible, the most realistic model of frictionless unilateral contact in the framework of nonhomogeneous linear elasticity and could be extended straightforwardly to the anisotropic elasticity. The numerical experiments show the convergence of the adaptive strategy and are in accordance with the theoretical results.

The outline of the paper is as follows. We consider the variational formulations and prove well posedness results in Section 2. In section 3, we describe the discrete problems, we prove a priori estimates and we perform the convergence analysis. In Section 4, we perform the a posteriori analysis, we introduce the residual error indicators and prove upper and lower bounds for the error in an appropriate norm. The details of the implementation and some numerical results are given in Section 5. The appendix, is devoted to the construction of the appropriate quasi-interpolant operator.

2. Variational formulation and discrete problems

Let Ω be a bounded domain of $\mathbb{R}^2$ with smooth boundary Γ, and $(\gamma_i)_i$, $1 \leq i \leq I$, a given number of Lipschitz continuous curves in Ω without self-intersections, such that $\Omega \setminus \bigcup_{i=1}^{I} \gamma_i$ is connected. We assume given a family Ω_j, $1 \leq j \leq J$ of pairwise disjoint Lipschitzian open subsets which constitutes a non overlapping decomposition of Ω, and the family $(\gamma_i)_i$, $1 \leq i \leq I$ is made of (parts) of interfaces of such decomposition.

$$\Omega = \bigcup_{j=1}^{J} \Omega_j \cup (\bigcup_{i=1}^{I} \gamma_i) \tag{1}$$

Let α be a function which is equal to the constant α_j on each Ω_j, $1 \le j \le J$ (see figure 1). We define the two parameters

$$\alpha_{min} = \min_{1\le j\le J} \alpha_j \qquad \alpha_{max} = \max_{1\le j\le J} \alpha_j,$$

and we assume α_{min} positive.

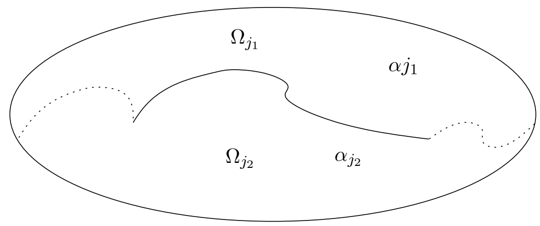

FIGURE 1. Zoom on the partition of the domain Ω. The plain curve is the contact zone γ

We denote by Ω_C the domain $\Omega \setminus (\bigcup_{i=1}^{I} \gamma_i)$, and we consider the problem

$$-\mathbf{div}\, \sigma = \mathbf{f} \qquad \text{in } \Omega_C, \tag{2}$$

$$\sigma = A^\alpha \varepsilon(\mathbf{u}) \qquad \text{in } \Omega_C, \tag{3}$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma\ , \tag{4}$$

$$[\mathbf{u}]\,\mathbf{n} \le 0, \qquad [\sigma_n] = 0, \qquad \sigma_n\,[\mathbf{u}]\,\mathbf{n} = 0 \qquad \text{on } \ \gamma_i,\ 1 \le i \le I \tag{5}$$

$$\sigma_n \le 0, \qquad \sigma_\tau = 0 \qquad \text{on}\, \gamma_i^{\pm},\ 1 \le i \le I. \tag{6}$$

Here $[\mathbf{u}] = \mathbf{u}{+} - \mathbf{u}^-$ denotes the jump in the displacement field across Γ_C, and the signs $\pm$ indicate the positive and negative directions with respect to the external normal $\mathbf{n}$. The unknown is the displacement field $\mathbf{u} = (u_1, u_2)$. The symmetric stress tensor $\sigma = (\sigma_{ij})$, $i, j = 1, 2$ is linked to the displacement by Hooke's law (3) where $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$ denotes the symmetric strain tensor. We set

$$A^\alpha = \alpha_j A \qquad \text{in } \Omega_j,\ 1 \le j \le J,$$

where A is the elasticity tensor. The body is subjected to volume forces $\mathbf{f}$. We have used the following standard notation

$$\sigma\mathbf{n} = \{\sigma_{ij}\, n_j\}_{i=1}^2\,,\ \sigma_n = \sigma_{ij}\, n_j\, n_i,\ \sigma_\tau = \sigma\mathbf{n} - \sigma_n\,\mathbf{n} = \left\{\sigma_\tau^i\right\}_{i=1}^2,$$
$$\{A^\alpha\varepsilon(\mathbf{u})\}_{ij} = A^\alpha_{ijk\ell}\varepsilon_{k\ell},\ A^\alpha_{ijk\ell} = A^\alpha_{jik\ell} = A^\alpha_{k\ell ij},\ A^\alpha_{ijk\ell} \in L^\infty(\Omega).$$

The fourth order tensor A^α satisfies the ellipticity condition

$$A^\alpha_{ijk\ell}\xi_{ji}\xi_{k\ell} \ge a_0|\xi|^2,\ \forall \xi_{ji} = \xi_{ij},\ a_0 > 0. \tag{7}$$

We use the summation convention over repeated indices.

2.1. **Variational formulation and well-posedness.** We assume that the data $\mathbf{f}$ are in $L^2(\Omega_C)$ and we introduce the Sobolev space

$$H^1_\Gamma(\Omega_C)^2 = \left\{\mathbf{v} = (v_1, v_2) \in H^1(\Omega_C)^2;\ v_i = 0 \text{ on}\, \Gamma,\ i = 1, 2\right\},$$

and the closed convex set (of admissible displacements)

$$K_C = \left\{\mathbf{v} = (v_1, v_2) \in H^1_\Gamma(\Omega_C)^2;\ [\mathbf{v}]\,\mathbf{n} \le 0 \text{ a.e on}\, \Gamma_C\right\}. \tag{8}$$

Problem (2)-(6) admits the following equivalent variational form:

$$\left\{ \begin{array}{l} \text{find}\,\mathbf{u} \in K_C \text{ such that,} \\ \int_{\Omega_C} \sigma(\mathbf{u}) : \varepsilon(\mathbf{v}-\mathbf{u})\,d\mathbf{x} \geq \int_{\Omega_C} \mathbf{f}(\mathbf{v}-\mathbf{u})\,d\mathbf{x},\ \forall \mathbf{v} \in K_C. \end{array} \right. \tag{9}$$

Due to the boundedness of α, the bilinear form on the right-hand side is continuous, and the positivity of α_{min} ensures its coercivity thanks to the Korn inequality which holds in Ω_C. The following result is a consequence of Stampacchia theorem [29]

Proposition 2.1. *For any data* $\mathbf{f} \in L^2(\Omega_C)^2$*, problem* (9) *admits a unique solution* $\mathbf{u} \in K_C$.

The equivalence between problem (2)-(6) and the variational formulation (9) as well as the precise mathematical meaning of the boundary, respectively the contact condition on Γ_C, requires some care. Let Σ_i be one of $\partial\Omega_{j_{1,i}}$ or $\partial\Omega_{j_{2,i}}$ (the subdomains that share the interface γ_i, and that we denote for brevity Ω_i). The space $H^{\frac{1}{2}}(\Sigma_i)$ for Lipschitz closed curves Σ_i is defined in [26, Chapter1, relation (1.16)]. For the convenience of the reader, we recall that its norm is

$$\|\mathbf{v}\|^2_{H^{\frac{1}{2}}(\Sigma_i)} = \|\mathbf{v}\|^2_{L^2(\Sigma_i)} + \int_{\Sigma_i}\int_{\Sigma_i} \frac{|\mathbf{v}(x)-\mathbf{v}(y)|^2}{|x-y|^2}\,ds(x)\,ds(y).$$

We denote by $H^{-\frac{1}{2}}(\Sigma_i)$ its dual. For $\sigma \in L^2(\Omega_i)^4$, $\operatorname{div}\sigma \in L^2(\Omega_i)^2$, the traces $(\sigma.\mathbf{n}_i)^{\pm}$ can be defined as elements of $H^{-\frac{1}{2}}(\Sigma_i)$, and the trace operator with values in $H^{-\frac{1}{2}}(\Sigma_i)$ is continuous [26]. We denote by $H^{\frac{1}{2}}_{00}(\gamma_i)$ the subspace of $H^{\frac{1}{2}}(\Sigma_i)$ consisting of functions with support in $\bar{\gamma}_i$. Note that the definition of $H^{\frac{1}{2}}_{00}(\gamma_i)$ does not depend on the choice of Σ_i being the boundary of Ω_{j_1} or Ω_{j_2}. This space is defined for smooth γ_i in [42, Chapter 1]. When γ_i are smooth enough (say $C^{1,1}$), we obtain, by standard arguments, the precise interpretation of conditions (5)-(6): for each i, $1 \leq i \leq I$,

$$< \sigma_n(\mathbf{u})^- - \sigma_n(\mathbf{u})^+, \varphi >_{\gamma_i} = 0, \qquad \forall \varphi \in H^{\frac{1}{2}}_{00}(\gamma_i), \tag{10}$$

$$< \sigma_n(\mathbf{u}), \varphi >_{\gamma_i} \leq 0, \qquad \forall \varphi \in H^{\frac{1}{2}}_{00}(\gamma_i),\ \varphi \geq 0, \tag{11}$$

$$< \sigma_\tau(\mathbf{u}), \varphi >_{\gamma_i} = 0, \qquad \forall \varphi = (\varphi_1, \varphi_2) \in H^{\frac{1}{2}}_{00}(\gamma_i),\ \varphi \cdot \mathbf{n} = 0, \tag{12}$$

and

$$< \sigma_n(\mathbf{u})\,[\mathbf{u}]\,\mathbf{n}, \varphi >_{\gamma_i} = 0, \qquad \forall \varphi \in H^{\frac{1}{2}}_{00}(\gamma_i), \tag{13}$$

where $< .,. >_{\gamma_i}$ denotes the duality product between $H^{\frac{1}{2}}_{00}(\gamma_i)$ and its dual $H^{-\frac{1}{2}}(\gamma_i)$ (we will not distinguish between scalar and vector valued cases).

2.2. **Hybrid variational formulation.** We introduce the spaces

$$\mathbf{V}(\Omega_j) = \left\{\mathbf{v} \in H^1(\Omega_j),\ \mathbf{v} = 0 \text{ on } \Gamma \cap \partial\Omega_j\right\}, \quad 1 \leq j \leq J,$$

and the space

$$\mathbf{V} = \left\{\mathbf{v} \in L^2(\Omega); \mathbf{v}_{|\Omega_j} \in \mathbf{V}(\Omega_j)\right\} \cap H^1(\Omega_C).$$

We define the Lagrange multiplier convex set

$$M = \{\mu = (\mu_i) \in \prod_{i=1}^{I} H^{-\frac{1}{2}}(\gamma_i),\ \sum_{i=1}^{I} < \mu_i, \psi_i >_{\gamma_i} \geq 0,$$
$$\forall \psi = (\psi_i) \in \prod_{i=1}^{I} H_{00}^{\frac{1}{2}}(\gamma_i),\ \psi_i \geq 0, 1 \leq i \leq I\}.$$

We denote

$$\|\mu\|_{-\frac{1}{2},*} = (\sum_{i=1}^{I} \|\mu_i\|^2_{H^{-\frac{1}{2}}(\gamma_i)})^{\frac{1}{2}}.$$

In order to obtain error estimates which are independent of α, we will work with the norm

$$\|\mathbf{v}\|_\alpha = \big(\sum_{j=1}^{J} |\alpha_j^{\frac{1}{2}} \mathbf{v}|^2_{H^1(\Omega_j)^2} + \|\mathbf{v}\|^2_{L^2(\Omega_j)^2}\big)^{\frac{1}{2}}. \tag{14}$$

We introduce the following notations : for $\mathbf{u} = (\mathbf{u}^j)_{1 \leq j \leq J}$, $\mathbf{v} = (\mathbf{v}^j)_{1 \leq j \leq J}$ in $\mathbf{V}$ and μ in $\prod_{i=1}^{I} H^{-\frac{1}{2}}(\gamma_i)$

$$a_\alpha(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{J} \int_{\Omega_j} (A^\alpha)_{mnkh} \varepsilon^j_{mn}(\mathbf{u}^j)\, \varepsilon^j_{kh}(\mathbf{v}^j)\, d\mathbf{x},$$

$$(\mathbf{f}, \mathbf{v}) = \sum_{j=1}^{J} \int_{\Omega_j} \mathbf{f}^j \cdot \mathbf{v}^j\, d\mathbf{x},$$

and

$$b(\mu, \mathbf{v}) = \sum_{i=1}^{I} < \mu_i, \mathbf{v}^{j_{1,i}} \cdot \mathbf{n}^{j_{1,i}} + \mathbf{v}^{j_{2,i}} \cdot \mathbf{n}^{j_{2,i}} >_{\gamma_i} = \sum_{i=1}^{I} < \mu, [\mathbf{v}]\, \mathbf{n}^i >_{\gamma_i} .$$

The hybrid variational formulation of problem (9) consists of finding $\mathbf{u} \in \mathbf{V}$ and $\lambda \in M$, such that

$$\left\{ \begin{array}{l} a_\alpha(\mathbf{u}, \mathbf{v}) + b(\lambda, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{V}, \\ b(\mu - \lambda, \mathbf{u}) \leq 0, \qquad \forall \mu \in M. \end{array} \right. \tag{15}$$

The existence and uniqueness of the solutions of problem (15) follow, in a standard way, from the ellipticity of $a_\alpha(.,.)$ and the usual Brezzi-Babuska inf-sup condition on $b(.,.)$ (see [29], III. Theorem 9.4)

Proposition 2.2. *Problem* (15) *admits a unique solution* $(\mathbf{u}, \lambda) \in \mathbf{V} \times M$. *Moreover, we have*

$$\lambda = (\lambda_i)_{1 \leq i \leq I} = -(\sigma_{\ell m})^{j_{1,i}} n_\ell^{j_{1,i}} n_m^{j_{1,i}} = -(\sigma_{\ell m})^{j_{2,i}} n_\ell^{j_{2,i}} n_m^{j_{2,i}}. \tag{16}$$

3. The discrete problems

We assume that each of the subdomains Ω_j, $1 \leq j \leq J$ is polygonal and the interfaces γ_i, $1 \leq i \leq I$ are straight lines . More general cases require additional and non-essential technicalities that we omit for simplicity. We consider regular families $(\mathcal{T}_h^j)_h$ of partitions of Ω_j, $1 \leq j \leq J$, into a finite number of triangles which satisfy the usual admissibility conditions [18].

We denote by $h_j = \max_{K\in\mathcal{T}_h^j} h_K$ the discretization parameter on Ω_j, and $h = \max(h_j)_{1\le j\le J}$. We also assume that the endpoints of each γ_i, $\mathbf{c}_1^i$ and $\mathbf{c}_2^i$ are common vertices of the triangulations $\mathcal{T}_h^{j_{1,i}}$ and $\mathcal{T}_h^{j_{2,i}}$, and the traces of the triangulations $\mathcal{T}_h^{j_{1,i}}$ and $\mathcal{T}_h^{j_{2,i}}$ on γ_i are one-dimensional triangulations that we denote by $\mathcal{T}_{h,\gamma_i}^{\ell}$, $\ell = 1,2$, $1\le i\le I$. The set of vertices of $\mathcal{T}_{h,\gamma_i}^{\ell}$, $\ell=1,2$ is denoted by

$$\zeta_{h,i}^{\ell} = \left\{\mathbf{c}_1^i = \mathbf{x}_{0,i}^{\ell}, \mathbf{x}_{1,i}^{\ell}, \ldots, \mathbf{x}_{m_\ell,i}^{\ell} = \mathbf{c}_2^i\right\},$$

and their elements are $t_{k,i}^{\ell} = \left]\mathbf{x}_{k,i}^{\ell}, \mathbf{x}_{k+1,i}^{\ell}\right[$, $0\le k\le m_\ell - 1$. We consider the (affine) finite element spaces

$$\mathbf{V}_h(\Omega_j) = \left\{\mathbf{v}_h^j \in \mathcal{C}(\overline{\Omega_j}),\ \forall K\in\mathcal{T}_h^j\ \mathbf{v}_{h|K}^j \in \mathcal{P}_1(K)^2,\ \mathbf{v}_h^j = 0 \text{ on } \Gamma\right\},$$

and the space

$$\mathbf{V}_h = \left\{\mathbf{v}_h \in L^2(\Omega); \mathbf{v}_{h|\Omega_j} \in \mathbf{V}_h(\Omega_j)\right\} \cap H^1(\Omega_C).$$

Note that $\mathbf{V}_h \subset \mathbf{V}$.

For the approximation of the Lagrange multipliers, for each $1\le i\le I$, and $\ell = 1,2$ we introduce the affine, respectively piecewise, finite element spaces

$$\begin{aligned} W_h^{1,\ell}(\gamma_i) &= \left\{\mu_h \in \mathcal{C}(\overline{\gamma}_i),\ \exists \mathbf{v}_h \in \mathbf{V}_h \text{ such that } \mathbf{v}_{h|\Omega_{j_{\ell(i)}}} \cdot \mathbf{n}^{j_{\ell(i)}} = \mu_h \text{ on } \gamma_i\right\}, \\ &= \big\{\mu_h \in \mathcal{C}(\overline{\gamma}_i),\ \mu_{h|t_{k,i}} \in \mathcal{P}_1(t_{k,i}), 1\le k\le m_\ell - 2 \\ &\qquad \mu_{h|t_{0,i}} \in \mathcal{P}_0(t_{0,i}),\ \mu_{h|t_{m_\ell-1,i}} \in \mathcal{P}_0(t_{m_\ell-1,i})\big\}, \end{aligned}$$

and the discrete convex cones

$$M_h^{1,\ell}(\gamma_i) = \left\{\mu_h \in W_h^{1,\ell}(\gamma_i), \mu_h \ge 0 \text{ on } \gamma_i\right\},$$

$$M_h^{1,\ell,*}(\gamma_i) = \left\{\mu_h \in W_h^{1,\ell}(\gamma_i), \int_{\gamma_i} \mu_h\,\psi_h \ge 0\ \forall \psi_h \in M_h^{1,\ell}(\gamma_i)\right\}.$$

These two convex cones are commonly used to express the jump conditions on the contact zone. The choice of one is linked to the way one selects to enforce the nonnegativity conditions (either on the displacement field or on the normal component of the stress tensor). We set $M_h(\gamma_i) = M_h^{1,\ell}(\gamma_i)$ or $M_h^{1,\ell,*}(\gamma_i)$, $\ell=1,2$ and we define

$$W_h^1 = \prod_{i=1}^{I} W_h^{1,\ell}(\gamma_i), \qquad M_h = \prod_{i=1}^{I} M_h(\gamma_i).$$

Note that if $M_h(\gamma_i) = M_h^{1,\ell}(\gamma_i)$ then $M_h \subset M$, $M_h^{1,\ell}(\gamma_i) \subset M_h^{1,\ell,*}(\gamma_i)$ but if $M_h(\gamma_i) = M_h^{1,\ell,*}$ then $M_h \not\subset M$.

The discrete problem reads : find $(\mathbf{u}_h, \lambda_h) \in \mathbf{V}_h \times M_h$ such that

$$\left\{\begin{array}{l} a_\alpha(\mathbf{u}_h, \mathbf{v}_h) + b(\lambda_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mu_h - \lambda_h, \mathbf{u}_h) \le 0, \qquad \forall \mu_h \in M_h. \end{array}\right. \tag{17}$$

The $\mathbf{V}_h$-ellipticity of the bilinear form holds thanks to the Korn inequality, still valid in the case of unilateral contact cracks. It is also readily checked that

$$\{\mu_h \in M_h, \quad b(\mu_h, \mathbf{v}_h) = 0,\ \forall \mathbf{v}_h \in \mathbf{V}_h\} = \{0\}.$$

Therefore, the next proposition follows from standard saddle-point theory in the finite dimensional setting.

Proposition 3.1. *With M_h based on the local choices $M_h(\gamma_i) = M_h^{1,\ell}(\gamma_i)$, or $M_h^{1,\ell,*}(\gamma_i)$, $\ell = 1,2$, $1 \le i \le I$, there exists a unique solution $(\mathbf{u}_h, \lambda_h)$ of the discrete problem* (17).

In all what follows we denote by C all the positives constants independent of α and h.

In order to obtain optimal approximation properties, we need the bilinear form $b(.,.)$ to satisfy a uniform inf-sup condition (with respect to h). This condition requires some assumptions on the triangulations on each γ_i, $1 \le i \le I$. The simplest and sufficient one is the quasi-uniformity, however, we do not make this assumption which is too stringent especially for adaptive mesh refinement which is one of the objectives of this work. Instead, we will make the following assumption: the 1D triangulations $\mathcal{T}^\ell_{h,\gamma_i}$, $\ell = 1,2$, satisfy the Crouzeix-Thomée criterion [22]

$$\frac{|t^\ell_{k,i}|}{|t^\ell_{k',j}|} \le C\beta^{|k-k'|}, \quad \forall k, k' \ (0 \le k, k' \le m_\ell - 1), \tag{18}$$

where $1 \le \beta \le 4$.

Remark 3.2. *Meshes satisfying Assumption 1 allows the adaptive mesh refinement unlike the quasi-uniform meshes* [9]

We recall some approximation tools which will be used in the following analysis. Let R_h^j, and r_h^i, be the Lagrange interpolation operators with values in $\mathbf{V}_h(\Omega_j)$ $(1 \le j \le J)$ and $W_h^{1,\ell}(\gamma_i)$ $(\ell = 1,2)$, $1 \le i \le I$, respectively. There exists a constant $C > 0$, such that $\forall \mathbf{v} \in (H^2(\Omega_j))^2$ and $v \in H^{\frac{3}{2}}(\gamma_i)$ ([18]) the following estimates hold

$$\|\mathbf{v} - R_h^j \mathbf{v}\|_{(H^1(\Omega_j))^2} \le Ch\|\mathbf{v}\|_{(H^2(\Omega_j))^2} \quad \text{and} \quad \|v - r_h^i v\|_{L^2(\gamma_i)} \le Ch^{\frac{3}{2}}\|v\|_{H^{\frac{3}{2}}(\gamma_i)}. \tag{19}$$

Let us denote by γ any γ_i, $1 \le i \le I$. We define the projection operator π_h^1 : $L^2(\gamma) \mapsto W_h^{1,\ell}(\gamma)$, with respect to the scalar product in $L^2(\gamma)$, which satisfies the following properties (see [11], [9]). Given $\mu \in [0,1]$ and $\nu \in \left]\frac{1}{2}, 2\right]$, there exists a constant $c > 0$ which is independent of h, such that for all functions $\varphi \in H^\nu(\gamma)$,

$$\|\varphi - \pi_h^1\varphi\|_{H^{-\mu}(\gamma)} + h^{\mu+\frac{1}{2}}\|\varphi - \pi_h^1\varphi\|_{H^{\frac{1}{2}}(\gamma)} \le ch^{\mu+\nu}\|\varphi\|_{H^\nu(\gamma)}. \tag{20}$$

Proposition 3.3. *Under assumption* (18) *on the triangulations $\mathcal{T}^\ell_{h,\gamma_i}$, $\ell = 1$ or 2, $1 \le i \le I$, the following inf-sup condition holds*

$$\inf_{\mu_h \in W_h^1} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mu_h, \mathbf{v}_h)}{\|\mu_h\|_{H^{-\frac{1}{2}}}\|\mathbf{v}_h\|_\alpha} \ge \delta\,(\frac{\alpha_m}{\alpha_M})^{\frac{1}{2}}. \tag{21}$$

The constant δ is independent of h and α.

Proof. Let $\mu_h \in W_h^{1,\ell}$. We want to construct $\mathbf{v}_h \in \mathbf{V}_h$ satisfying

$$b(\mu_h, \mathbf{v}_h) \ge c_1(\alpha)\|\mu_h\|^2_{-\frac{1}{2},*} \qquad \text{and} \qquad c_2(\alpha)\|\mathbf{v}_h\|_\alpha \le \|\mu_h\|_{-\frac{1}{2},*}, \tag{22}$$

where $c_\ell(\alpha)$, $\ell = 1,2$ are constants depending, on α but not on h. Let us consider $\mathbf{v}$, the solution of the problem

$$\begin{cases} \operatorname{div} \alpha(x)\nabla\,\mathbf{v} = 0, \text{ in } \Omega_C \\ \alpha(x)\nabla\,\mathbf{v}\cdot\mathbf{n} = 0, \text{ on } \Gamma \\ \alpha(x)\nabla\,\mathbf{v}\cdot\mathbf{n}^\pm = \mu_{h,i}\,\mathbf{n}^\pm, \text{ on } \gamma_i, \ 1 \le i \le I. \end{cases}$$

The existence of such a $\mathbf{v}$ comes from the direct method of the calculus of variations. It is clear that $\mathbf{v}$ is also the solution of the equivalent variational problem

$$\int_{\Omega_C} \alpha(x)\,\nabla\,\mathbf{v}\,\nabla\,\mathbf{w}\,dx = b(\mu_h, \mathbf{w}), \qquad \forall \mathbf{w} \in H^1(\Omega_C)^2,$$

and satisfies, by the usual stability inequality and the continuous inf-sup condition on $b(.,.)$,

$$c^- \,\|\mu_h\|_{-\frac{1}{2},*} \leq \|\mathbf{v}\|_\alpha \leq c^+ \|\mu_h\|_{-\frac{1}{2},*}, \tag{23}$$

where c^+ and c^- are constants independent of h and α.

We set $\mathbf{v}_h \in \mathbf{V}_h$ such that $\left[\mathbf{v}_{h|\gamma_i}\mathbf{n}\right] = \pi_h^1(\left[\mathbf{v}_{|\gamma_i}\mathbf{n}\right])$, $1 \leq i \leq I$ and

$$\begin{aligned} \|\mathbf{v}_h\|_\alpha \leq c\,\alpha_M^{\frac{1}{2}}\,\|\mathbf{v}_h\|_{H^1(\Omega_C)} &\leq c\,\alpha_M^{\frac{1}{2}} \sum_{i=1}^{I} \|\pi_h^1 \left[\mathbf{v}_{|\gamma_i}\mathbf{n}\right]\|_{H^{-\frac{1}{2}}(\gamma_i)} \\ &\leq c\,\alpha_M^{\frac{1}{2}} \sum_{i=1}^{I} \|\left[\mathbf{v}_{|\gamma_i}\mathbf{n}\right]\|_{H^{-\frac{1}{2}}(\gamma_i)} \leq c\,\alpha_M^{\frac{1}{2}}\,\alpha_m^{-\frac{1}{2}}\,\|\mathbf{v}\|_\alpha. \end{aligned} \tag{24}$$

Such a $\mathbf{v}_h$ is built using a stable finite element extension operator similar to the standard local regularization operator studied in [10]. Next, we note that

$$b(\mu_h, \mathbf{v}_h) = b(\mu_h, \mathbf{v}) = \|\mathbf{v}\|_\alpha^2 \geq (c^-)^2\,\|\mu_h\|_{-\frac{1}{2},*}^2,$$

and thanks to (23)-(24) and the trace theorem we have the second statement of (22). □

3.1. **A Remark on the use of piecewise constant Lagrange multipliers.** The discrete Lagrange multiplier space can also be defined with piecewise constants instead of the one dimensional affine finite elements: for $1 \leq i \leq I$,

$$W_h^{0,\ell}(\gamma_i) = \left\{\mu_h, \mu_{h|t_{k,i}^\ell} \in \mathcal{P}_0(t_{k,i}^\ell), 0 \leq k \leq m_{\ell,i} - 1\right\},$$

which yields the discrete convex cone

$$M_h^{0,\ell}(\gamma_i) = \left\{\mu_h \in W_h^{0,\ell}(\gamma_i), \mu_{h,i} \geq 0\right\}.$$

Denoting by $M_h^0 = \prod_{i=1}^I M_h^{0,\ell}(\gamma_i)$, and $W_h^0 = \prod_{i=1}^I W_h^{0,\ell}(\gamma_i)$, we have $M_h^0 \subset M$ and the resulting discrete problem also admits a unique solution $(\mathbf{u}_h, \lambda_h)$. However, this choice in the present case leads to non optimal approximation properties because of the presence of spurious modes. Following [9], we use, in this case, a stabilization technique by adding two bubble functions on each γ_i, $1 \leq i \leq I$

$$\varphi_{t_{k,i}^\ell}(\mathbf{x}) = \frac{6}{|t_{k,i}^\ell|}\lambda_1(\mathbf{x})\lambda_2(\mathbf{x}), \qquad \forall \mathbf{x} \in K_k,\ i = 0, m_{\ell,i} - 1,$$

where K_k, is the triangle having $t_{k,i}^\ell$, $k = 0$ or $m_{\ell,i} - 1$ as an edge and λ_1, λ_2 the barycentric coordinates associated to the vertices of $t_{k,i}^\ell$. We then replace $\mathbf{V}_h$ by

$$\widetilde{\mathbf{V}}_h = \mathbf{V}_h \oplus (\oplus_{i=1}^I \oplus_{t_{k,i}^\ell \in \mathcal{T}_{h,\gamma_i}^\ell} \mathbb{R}\varphi_{t_{k,i}^\ell}).$$

We denote by π_h^0 the L^2-projection operator $L^2(\gamma) \longrightarrow W_h^{0,\ell}(\gamma)$, defined as follows:

$$\int_\gamma v\psi_h\,d\sigma = \int_\gamma \pi_h^0(v)\psi_h\,d\sigma, \quad \forall \psi_h \in W_h^{0,\ell}(\gamma), \tag{25}$$

where π_h^0 satisfies the following estimates (see [9]). Namely, for the functions $\varphi \in H^\nu(\gamma)$, with $\nu = \frac{1}{2}$, or with $\nu = 1$, there exists a constant $c > 0$ independent of h such that

$$\|\varphi - \pi_h^0\varphi\|_{L^2(\gamma)} \le ch^\nu \|\varphi\|_{H^\nu(\gamma)}. \tag{26}$$

Moreover, if $\varphi \in L^2(\gamma)$, then

$$\|\varphi - \pi_h^0\varphi\|_{H^{-\frac{1}{2}}(\gamma)} \le ch^{\frac{1}{2}} \|\varphi - \pi_h^0\varphi\|_{L^2(\gamma)}. \tag{27}$$

The uniform inf-sup condition is obtained similarly to proposition 3.3

Proposition 3.4. *Under assumption* (18) *on the triangulations $\mathcal{T}^\ell_{h,\gamma_i}$, $\ell = 1$ or 2, $1 \le i \le I$, the following inf-sup condition holds*

$$\inf_{\mu_h \in W_h^0} \sup_{\mathbf{v}_h \in \widetilde{\mathbf{V}}_h} \frac{b(\mu_h, \mathbf{v}_h)}{\|\mu_h\|_{-\frac{1}{2},*} \|\mathbf{v}_h\|_\alpha} \ge \delta \left(\frac{\alpha_m}{\alpha_M}\right)^{\frac{1}{2}} > 0, \tag{28}$$

where the constant δ is independent of h or α.

3.2. **Convergence Analysis.** The convergence analysis and the a priori error estimates come from the standard approximation theory for unilateral contact problems [25, 29, 20, 41, 9, 7, 44, 35]. For the specific case of unilateral cracks we refer the reader to [46, 6, 47]. We give a brief summary for the reader convenience, we skip the proofs which are rather long but standard adaptations of arguments in [20, 9, 35] for the homogeneous case. We start with the following lemmas,

Lemma 3.5. *Let $(\mathbf{u}, \lambda)$ be the solution of problem* (15) *and $(\mathbf{u}_h, \lambda_h)$ the solution of problem* (17)*. Then the following estimate holds: for any $(\mathbf{v}_h, \mu_h) \in \mathbf{V}_h \times M_h$,*

$$\begin{aligned} a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) \le \big(a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) + b(\lambda - \mu_h, \mathbf{u}_h - \mathbf{u}) + b(\lambda - \lambda_h, \mathbf{u} - \mathbf{v}_h) \\ + b(\lambda - \mu_h, \mathbf{u}) + b(\lambda_h, \mathbf{u})\big). \end{aligned} \tag{29}$$

From (29) and standard computations, we deduce

Lemma 3.6. *Let $(\mathbf{u}, \lambda)$ be the solution of problem* (15)*. Assume that $\mathbf{u}_{|\Omega_j} \in H^2(\Omega_j)$, $1 \le j \le J$. Let $(\mathbf{u}_h, \lambda_h)$ be the solution of problem* (17) *with $M_h(\gamma_i) = M_h^{1,\ell,*}(\gamma_i)$ or $M_h(\gamma_i) = M_h^{\ell,0}(\gamma_i)$, $1 \le i \le I$. Then the following estimate holds*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_\alpha \le C\Big(\Big\{\sum_{j=1}^J \alpha_j^{\frac{1}{2}}\, h_j \|\mathbf{u}\|_{H^2(\Omega_j)^2}\Big\} \|\lambda - \lambda_h\|_{-\frac{1}{2},*} \\ + \Big\{\sum_{j=1}^J \alpha_j^{\frac{1}{2}}\, h_j^{\frac{3}{4}} \|\mathbf{u}\|_{H^2(\Omega_j)^2}\Big\}\Big). \end{aligned} \tag{30}$$

Remark 3.7. *It is well known that a priori error estimates are not optimal if the number of points where the constraint changes from binding to non binding* [29, 14, 15]*. In 2D (which is the case in this article), if $M_h(\gamma_i) = M_h^{1,\ell,*}(\gamma_i)$, and the number of points on each γ_i where the constraint changes from binding to non* *binding, is finite, $1 \leq i \leq I$, the following (optimal) estimate holds*

$$\|\mathbf{u}-\mathbf{u}_h\|_\alpha \leq C\Big(\left\{\sum_{j=1}^{J} \alpha_j^{\frac{1}{2}}\, h_j \|\mathbf{u}\|_{H^2(\Omega_j)^2}\right\} \|\lambda-\lambda_h\|_{-\frac{1}{2},*} + \left\{\sum_{j=1}^{J} \alpha_j^{\frac{1}{2}}\, h_j \|\mathbf{u}\|_{H^2(\Omega_j)^2}\right\}\Big). \tag{31}$$

Remark 3.8. *Note that the regularity assumption is local and thus generally is satisfied. The cases for which such an assumption is not valid arises only when some subdomains Ω_j contain corners or if some changes in the boundary conditions (from Neumann to Dirichlet) occur. We exclude such situations in order to focus on the contact zone.*

The proof of the following result is standard (see [20] for example)

Lemma 3.9. *Under the same assumptions as in the previous lemma, the following estimate holds*

$$(\frac{\alpha_m}{\alpha_M})^{\frac{1}{2}} \|\lambda-\lambda_h\|_{-\frac{1}{2},*} \leq C(\|\mathbf{u}-\mathbf{u}_h\|_\alpha + \left\{\sum_{j=1}^{J} \alpha_j\, h_j^2 \|\mathbf{u}\|^2_{H^2(\Omega_j)^2}\right\}^{\frac{1}{2}}), \tag{32}$$

with a constant C independent of α.

Remark 3.10. *The choice of the multiplier spaces yields the same result in each case. However, in the case of smooth solutions, namely $\lambda \in H^s(\gamma_i)$, $1 \leq i \leq I$, $s > \frac{1}{2}$, the approximation order for piecewise constant multipliers does not change, while it increases for the other choices.*

Finally, assembling the estimates of the two previous lemmas, we deduce

Theorem 3.11. *Let $(\mathbf{u}, \lambda)$ be the solution of problem (15). Assume that $\mathbf{u}_{|\Omega_j} \in H^2(\Omega_j)$, $1 \leq j \leq J$. Let $(\mathbf{u}_h, \lambda_h)$ be the solution of problem (17) with $M_h = M_h^{\ell,*}$ or $M_h = M_h^0$, the following estimate holds*

$$\|\mathbf{u}-\mathbf{u}_h\|_\alpha + (\frac{\alpha_m}{\alpha_M})^{\frac{1}{2}} \|\lambda-\lambda_h\|_{-\frac{1}{2},*} \leq C \left\{\sum_{j=1}^{J} \alpha_j^{\frac{1}{2}}\, h_j^{\frac{3}{4}} \|\mathbf{u}\|_{H^2(\Omega_j)^2}\right\}. \tag{33}$$

Remark 3.12.

ii) *In 2D (our case), if $M_h(\gamma_i) = M_h^{1,\ell,*}(\gamma_i)$, and the number of points on each γ_i where the constraint changes from binding to non binding, is finite,$1 \leq i \leq I$, the following estimate holds*

$$\|\mathbf{u}-\mathbf{u}_h\|_\alpha + (\frac{\alpha_m}{\alpha_M})^{\frac{1}{2}} \|\lambda-\lambda_h\|_{-\frac{1}{2},*} \leq C \left\{\sum_{j=1}^{J} \alpha_j^{\frac{1}{2}}\, h_j \|\mathbf{u}\|_{H^2(\Omega_j)^2}\right\}. \tag{34}$$

Remark 3.13.
1. *In the case of homogeneous media, we retrieve the expected rate of convergence of $O(h)$.*
2. *The case $M_h(\gamma_i) = M_h^{1,\ell}(\gamma_i)$ yields the rate of convergence $O(\alpha^{\frac{1}{2}} h^{\frac{1}{2}})$ by adapting the argument of* [9].

4. A posteriori analysis

The a posteriori analysis by residual error indicators that we develop in this section aims to present, in a relatively general and unified framework, the most used approximation approaches (conforming and nonconforming) for contact problems. Since these two approximation methods resort to different techniques to perform the analysis and to derive the a posteriori estimates we will presents the nonconforming case in details and we refere to the literature for the conforming one (e.g. [17, 33]).

There are many difficulties to deal with in this analysis: discontinuous coefficients with high contrast, incompatible meshes at the interfaces of the subdomains, handling the contact conditions, and possible low regularity in these contact zones. In particular, a key point to conduct such an analysis is to build a Clément type operator: an interpolation operator for singular functions, that includes all these constraints and that satisfies optimal approximation properties. As usual, the analysis consists to define the well suited error indicators and to obtain upper and lower bounds of the error with them. In addition, we will give, as far as possible, an interpretation of the theoretical error estimates, in order to highlight how each error indicator acts in the adaptive process.

For the a posteriori error estimates we assume that $\mathbf{f} \in L^2(\Omega)^2$ and we fix $\mathbf{f}_h$ to be a finite element approximation of it associated with $\mathcal{T}_h$ (usually this approximation is the same as for the displacement).

Notation . *Given $K \in \mathcal{T}_h^j$, we denote by $\mathcal{E}_K$ the set of its edges not contained on the boundary $\partial\Omega_j$, $1 \le j \le J$. The union of all $\mathcal{E}_K$, $K \in \mathcal{T}_h^j$ is denoted by $\mathcal{E}_{h,j}$ and the union of $\mathcal{E}_{h,j}$, $1 \le j \le J$ will be denoted $\mathcal{E}_h$. We denote by $\mathcal{E}_{h,j,\gamma_i}$ the set of edges of $\mathcal{T}_h^j$ which are contained in γ_i and we set $\mathcal{E}_{h,\gamma_i} = \mathcal{E}_{h,j\ell(i),\gamma_i}$, $1 \le i \le I$, $\ell = 1$ or 2 with the same choice of ℓ as for M_h. With each edge $e \in \mathcal{E}_h$ or $e \in \mathcal{E}_{h,\gamma_i}$, $1 \le i \le I$, we associate a unit vector $\mathbf{n}_e$ normal to e and we denote by $[\varphi]_e$ the jump of the piecewise continuous (vector valued) function φ across e in the direction $\mathbf{n}_e$. For each $K \in \mathcal{T}_h$ we denote by h_K the diameter of K and we denote by h_e the diameter of e, $e \in \mathcal{E}_K$.*

Remark 4.1. *In all that follows when we write "contact zone" it means the zone where the contact is active, that is where the jump $(\mathbf{u}\,\mathbf{n})^+ - (\mathbf{u}\,\mathbf{n})^- = 0$. In practice, we know dynamically that zone during the computations(see Section.* (68)*).*

We define two kinds of residual error indicators

- An error indicator for the elements of the mesh. For each element $K \in \mathcal{T}_h$, we set

$$\eta_K = \alpha_K^{-\frac{1}{2}} h_K \|\mathbf{f}_h + \operatorname{div} \alpha\sigma(\mathbf{u}_h)\|_{L^2(K)^2} + \frac{1}{2} \sum_{e \in \mathcal{E}_K} \alpha_e^{-\frac{1}{2}} h_e^{\frac{1}{2}} \|[\alpha\sigma(\mathbf{u}_h)\,\mathbf{n}_e]_e\|_{L^2(e)^2}. \tag{35}$$

- Error indicators for the edges on the contact zone. For each $e \in \mathcal{E}_{h,\gamma_i}$, $1 \le i \le I$, we set

$$\eta_e = h_e^{-\frac{1}{2}} \|[\mathbf{u}_h\,\mathbf{n}_e]_e\|_{L^2(e)^2}, \tag{36}$$

  and

$$\eta_{C,e} = \alpha_e^{-\frac{1}{2}} h_e^{\frac{1}{2}} \|\beta_e\,\lambda_h + \mathbf{n}_e \cdot \alpha\sigma(\mathbf{u}_h)\,\mathbf{n}_e\|_{L^2(e)}, \tag{37}$$

with $\beta_e = 1$ if $M_h(\gamma_i) = M_h^0(\gamma_i)$ or $M_h^{1,\ell}(\gamma_i)$ and $\beta_e = 0$ if $M_h = M_h^{1,\ell,*}(\gamma_i)$. We denote indifferently the scalar product $x \cdot \alpha\sigma(\mathbf{u}_h)\, y$ or $\alpha\sigma(\mathbf{u}_h)\, x\, y$ for any vectors x, y.

Remark 4.2. *The residual error indicators appear from the computation of the residu of the equation in the appropriate norm (the dual of the energy norm see* [48]*). The first indicator η_K, is the standard one for a nonhomgeneous material in linear elasticity* [48]*. The second one, η_e is due to the nonconformity of the meshes at the contact zone. Basically, it is the jump in the $H^{\frac{1}{2}}$-norm of the normal component of the displacement. Note that it is zero, except at locations where the contact is active and where the meshes or not compatible. The last error indicator is specific to the contact condition it could be expressed in η_K -in this case it looks like a Neumann condition on each γ_i- but for the clarity and ease of implementation we define it separately.*

Remark 4.3. *The error indicator $\eta_{C,e}$ measures the norm $H^{-\frac{1}{2}}$ of the multiplier (the pressure on the contact zone). Its interpretation is the following : in the conforming case, it expresses the residual $\lambda_h = \mathbf{n}_e.(\alpha\sigma(u_h)\,\mathbf{n}_e)$. In the nonconforming case, it expresses the residual $\mathbf{n}_e.(\alpha\sigma(u_h)\,\mathbf{n}_e)$ (which should vanishes for the exact solution if the contact is active (the jump is zero) and if the contact is not active, the non penetration condition imply that the pressure (on each side) should be zero (for the exact solution) and so is the "jump".*

4.1. **Abstract upper bound for the error : the nonconforming case.** We set $M_h(\gamma_i) = M_h^{1,\ell,*}(\gamma_i)$ and we will derive an abstract bound of the error by the error indicators.

The ellipticity of $a_\alpha(.,.)$ on $\mathbf{V}$ gives

$$\|\mathbf{u} - \mathbf{u}_h\|_\alpha^2 \le C\, a_\alpha(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h). \tag{38}$$

Since $\mathbf{V}_h \subset \mathbf{V}$, choosing $\mathbf{v} = \mathbf{v}_h \in \mathbf{V}_h$ in the first equation of (15) and subtracting the first equation of (17), we obtain

$$a_\alpha(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\lambda - \lambda_h, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{39}$$

We set $\mathbf{w} = \mathbf{u} - \mathbf{u}_h$, and we fix an approximation $\mathbf{w}_h$ of $\mathbf{w}$ in $\mathbf{V}_h$, then we deduce from (38) and (39) that

$$\|\mathbf{u} - \mathbf{u}_h\|_\alpha^2 \le C\,(a_\alpha(\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h) - b(\lambda - \lambda_h, \mathbf{w}_h)).$$

Integrating by parts, inserting $\mathbf{f}_h$ and considering separately the cases where $e \in \mathcal{E}_K$ belongs to $\cup_{i=1}^{I} \gamma_i$ or not, yields

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}_h\|_\alpha^2 \le C\,\Big(\sum_{K\in\mathcal{T}_h} \Big(&\int_K (\mathbf{f}_h + \operatorname{div} \alpha\,\sigma(\mathbf{u}_h)) \cdot (\mathbf{w}-\mathbf{w}_h)\, d\mathbf{x} + \int_K (\mathbf{f}-\mathbf{f}_h)\cdot(\mathbf{w}-\mathbf{w}_h)\, d\mathbf{x} \\ - \frac{1}{2} \sum_{e\in\mathcal{E}_K} \int_e [\mathbf{n}_e \cdot \alpha\sigma(\mathbf{u}_h)]\; (\mathbf{w}-\mathbf{w}_h)\, d\tau\Big) &- \sum_{i=1}^{I} \sum_{e\in\mathcal{E}_{h,\gamma_i}} \int_e \mathbf{n}_e \cdot \alpha\sigma(\mathbf{u}_h)\; [\mathbf{w}-\mathbf{w}_h]\; d\tau \\ &- b(\lambda, \mathbf{w}-\mathbf{w}_h) - b(\lambda-\lambda_h, \mathbf{w}_h)\Big). \end{aligned} \tag{40}$$

Choosing $\mathbf{w}_h$ as a conforming approximation of $\mathbf{w}$ (The construction of such $\mathbf{w}_h$ is not obvious, it is given in proposition 4.7) yields

$$\begin{aligned}(41)\quad \|\mathbf{u}-\mathbf{u}_h\|_\alpha^2 \le C\,\big(\sum_{K\in\mathcal{T}_h}\big(\int_K (\mathbf{f}_h+\operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\cdot(\mathbf{w}-\mathbf{w}_h)\,d\mathbf{x}\\ +\int_K (\mathbf{f}-\mathbf{f}_h)\cdot(\mathbf{w}-\mathbf{w}_h)\,d\mathbf{x} - \frac{1}{2}\sum_{e\in\mathcal{E}_K}\int_e [\mathbf{n}_e\cdot\alpha\sigma(\mathbf{u}_h)]\;(\mathbf{w}-\mathbf{w}_h)\,d\tau\big)\\ -\sum_{i=1}^{I}\sum_{e\in\mathcal{E}_{h,\gamma_i}}\int_e \mathbf{n}_e\cdot\alpha\sigma(\mathbf{u}_h)\,\mathbf{n}_e\,[\mathbf{w}\cdot\mathbf{n}_e]\;d\tau - b(\lambda,\mathbf{w})\big).\end{aligned}$$

Note that

$$b(\lambda,\mathbf{w}) = b(\lambda,\mathbf{u}_h) = b(\lambda-\lambda_h,\mathbf{u}_h).$$

Moreover, for each $e\in\mathcal{E}_{h,\gamma_i}$, $1\le i\le I$,

$$\begin{aligned}\int_e (\lambda-\lambda_h)\;[\mathbf{u}_h\cdot\mathbf{n}]\;d\tau &\le \|\lambda-\lambda_h\|_{H^{-\frac12}(e)}\|\,[\mathbf{u}_h\cdot\mathbf{n}]\,\|_{H^{\frac12}(e)}\\ &\le C\,\|\lambda-\lambda_h\|_{H^{-\frac12}(e)}(h_e^{-\frac12}\,\|\,[\mathbf{u}_h\cdot\mathbf{n}]\,\|_{L^2(e)}),\end{aligned}$$

summing up for all $e\in\mathcal{E}_{h,\gamma_i}$ and taking the square root, we obtain

$$\begin{aligned}(42)\quad (\sum_{i=1}^{I}\sum_{e\in\mathcal{E}_{h,\gamma_i}}\int_e(\lambda-\lambda_h)\;[\mathbf{u}_h\cdot\mathbf{n}]\;d\tau)^{\frac12} \le C\,\|\lambda-\lambda_h\|_{-\frac12,*}\\ (\sum_{i=1}^{I}\sum_{e\in\mathcal{E}_{h,\gamma_i}} h_e^{-1}\,\|\,[\mathbf{u}_h\cdot\mathbf{n}]\,\|_{L^2(e)}^2)^{\frac12}.\end{aligned}$$

Next, invoking the inf-sup condition and (39), we have

$$\begin{aligned}(\frac{\alpha_m}{\alpha_M})^{\frac12}\|\lambda_h-\mu_h\|_{-\frac12,*} &\le \sup_{\mathbf{w}_h\in\mathbf{V}_h}\frac{b(\lambda_h-\mu_h,\mathbf{w}_h)}{\|\mathbf{w}_h\|_\alpha}\\ &\le \sup_{\mathbf{w}_h\in\mathbf{V}_h}\frac{b(\lambda_h-\lambda,\mathbf{w}_h)}{\|\mathbf{w}_h\|_\alpha}+\sup_{\mathbf{w}_h\in\mathbf{V}_h}\frac{b(\lambda-\mu_h,\mathbf{w}_h)}{\|\mathbf{w}_h\|_\alpha}\\ &\le \sup_{\mathbf{w}_h\in\mathbf{V}_h}\frac{a_\alpha(\mathbf{u}-\mathbf{u}_h,\mathbf{w}_h)}{\|\mathbf{w}_h\|_\alpha}+\sup_{\mathbf{w}_h\in\mathbf{V}_h}\frac{b(\lambda-\mu_h,\mathbf{w}_h)}{\|\mathbf{w}_h\|_\alpha}\\ &\le C\|\mathbf{u}-\mathbf{u}_h\|_\alpha + C'\inf_{\mu_h\in M_h}\|\lambda-\mu_h\|_{-\frac12,*}.\end{aligned}$$

Using the triangle inequality

$$\|\lambda-\lambda_h\|_{-\frac12,*}\le\|\lambda_h-\mu_h\|_{-\frac12,*}+\|\lambda-\mu_h\|_{-\frac12,*}$$

and combining it with (40), we obtain

(43)
$$\|\mathbf{u}-\mathbf{u}_h\|_\alpha(\|\mathbf{u}-\mathbf{u}_h\|_\alpha+(\frac{\alpha_m}{\alpha_M})^{\frac{1}{2}}\,\|\lambda-\lambda_h\|_{-\frac{1}{2},*}) \leq C\,\big(\sum_{K\in\mathcal{T}_h}(\int_K(\mathbf{f}_h+\operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\cdot(\mathbf{w}-\mathbf{w}_h)\,d\mathbf{x}$$
$$+\int_K(\mathbf{f}-\mathbf{f}_h)\cdot(\mathbf{w}-\mathbf{w}_h)\,d\mathbf{x}-\frac{1}{2}\sum_{e\in\mathcal{E}_K}\int_e[\mathbf{n}_e\cdot\alpha\sigma(\mathbf{u}_h)]\;(\mathbf{w}-\mathbf{w}_h)\,d\tau)$$
$$-\sum_{i=1}^{I}\sum_{e\in\mathcal{E}_{h,\gamma_i}}\int_e\mathbf{n}_e\cdot\alpha\sigma(\mathbf{u}_h)\,[\mathbf{w}]\;d\tau-b(\lambda-\lambda_h,\mathbf{u}_h)\big).$$

Using (42), Cauchy-Schwartz inequality, and dividing by $\|\mathbf{w}\|_\alpha$, except the last term that we divide by $(\frac{\alpha_m}{\alpha_M})^{\frac{1}{2}}\,\|\lambda-\lambda_h\|_{-\frac{1}{2},*}$, this gives

$$\|\mathbf{u}-\mathbf{u}_h\|_\alpha \leq C\,\big(\sum_{K\in\mathcal{T}_h}(\|\mathbf{f}_h+\operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\|_{L^2(K)^2}\,\frac{\|\mathbf{w}-\mathbf{w}_h\|_{L^2(K)}}{\|\mathbf{w}\|_\alpha}\tag{44}$$
$$+\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^2}\,\frac{\|\mathbf{w}-\mathbf{w}_h\|_{L^2(K)^2}}{\|\mathbf{w}\|_\alpha}+\frac{1}{2}\sum_{e\in\mathcal{E}_K}\|\,[\alpha\sigma(\mathbf{u}_h)\mathbf{n}_e]\,\|_{L^2(e)^2}\frac{\|\mathbf{w}-\mathbf{w}_h\|_{L^2(e)^2}}{\|\mathbf{w}\|_\alpha})$$
$$-\frac{\sum_{i=1}^{I}\sum_{e\in\mathcal{E}_{h,\gamma_i}}\int_e\alpha\sigma(\mathbf{u}_h)\mathbf{n}_e\,[\mathbf{w}]\;d\tau}{\|\mathbf{w}\|_\alpha})$$
$$+\;C'\big((\frac{\alpha_m}{\alpha_M})^{-\frac{1}{2}}(\sum_{i=1}^{I}\sum_{e\in\mathcal{E}_{h,\gamma_i}}h_e^{-1}\,\|\,[\mathbf{u}_h\cdot\mathbf{n}]\,\|^2_{L^2(e)})^{\frac{1}{2}}\big).$$

Finally, we obtain the abstract upper bound in the nonconforming case

Lemma 4.4. *The upper bound on the error in the nonconforming case is given by*

(45)
$$\|\mathbf{u}-\mathbf{u}_h\|_\alpha+(\frac{\alpha_m}{\alpha_M})^{\frac{1}{2}}\,\|\lambda-\lambda_h\|_{-\frac{1}{2},*} \leq C\,\big(\sum_{K\in\mathcal{T}_h}(\|\mathbf{f}_h+\mathit{div}\,\alpha\,\sigma(\mathbf{u}_h))\|_{L^2(K)^2}\,\frac{\|\mathbf{w}-\mathbf{w}_h\|_{L^2(K)}}{\|\mathbf{w}\|_\alpha}$$
$$+\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^2}\,\frac{\|\mathbf{w}-\mathbf{w}_h\|_{L^2(K)^2}}{\|\mathbf{w}\|_\alpha}+\frac{1}{2}\sum_{e\in\mathcal{E}_K}\|\,[\alpha\sigma(\mathbf{u}_h)\mathbf{n}_e]\,\|_{L^2(e)^2}\frac{\|\mathbf{w}-\mathbf{w}_h\|_{L^2(e)^2}}{\|\mathbf{w}\|_\alpha})$$
$$-\frac{\sum_{i=1}^{I}\sum_{e\in\mathcal{E}_{h,\gamma_i}}\int_e\alpha\sigma(\mathbf{u}_h)\mathbf{n}_e\,[\mathbf{w}]\;d\tau}{\|\mathbf{w}\|_\alpha})$$
$$+\;C'\big((\frac{\alpha_m}{\alpha_M})^{-\frac{1}{2}}(\sum_{i=1}^{I}\sum_{e\in\mathcal{E}_{h,\gamma_i}}h_e^{-1}\,\|\,[\mathbf{u}_h\cdot\mathbf{n}]\,\|^2_{L^2(e)})^{\frac{1}{2}}+\inf_{\mu_h\in M_h}\|\lambda-\mu_h\|_{-\frac{1}{2},*}\big).$$

For the convenience of the reader, we only give the upper bound in the conforming case

Lemma 4.5. *The upper bound on the error in the conforming case is given by*

(46)
$$\|\mathbf{u}-\mathbf{u}_h\|_\alpha + \|\lambda-\lambda_h\|_{-\frac{1}{2},*} \le C \sum_{K\in\mathcal{T}_h} \Big(\|\mathbf{f}_h + div\,\alpha\,\sigma(\mathbf{u}_h)\|_{L^2(K)^2} \frac{\|\mathbf{w}-\mathbf{w}_h\|_{L^2(K)^2}}{\|\mathbf{w}\|_\alpha}$$
$$+ \|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^2} \frac{\|\mathbf{w}-\mathbf{w}_h\|_{L^2(K)^2}}{\|\mathbf{w}\|_\alpha} + \frac{1}{2}\sum_{e\in\mathcal{E}_K} \|\,[\alpha\,\sigma(\mathbf{u}_h)\mathbf{n}_e]\,\|_{L^2(e)^2} \frac{\|\mathbf{w}-\mathbf{w}_h\|_{L^2(e)^2}}{\|\mathbf{w}\|_\alpha}\Big)$$
$$- \sum_{e\in\mathcal{E}_{h\Gamma_C}} \frac{(\int_e (\lambda_h + \mathbf{n}_e\cdot\alpha\,\sigma(\mathbf{u}_h)\,\mathbf{n}_e)\,[\mathbf{w}\cdot\mathbf{n}_e]\;d\tau}{\|\mathbf{w}\|_\alpha} + \frac{1}{\|\mu\|_{-\frac{1}{2},*}} b(\mu-\mu_h,\mathbf{u}_h).$$

Remark 4.6. *Comparing these two upper bounds* (46) *and* (45)*, we note that the fourth term in* (45) *(the nonconforming case) do not appears in the second inequality. In fact, it is the "variational crime", similar to the consistency error in the a priori analysis. The two last terms in both inequalities are different since they involve the contact conditions and thus are inherent to the discretization in this region and particularly to the choice of the Lagrange multipliers discrete cone.*

4.2. **Final upper bound for the error.** In this section, we will derive the upper-bound of the discretization error by the error indicators. The common factors in the error estimates (45) and (46) will be handled by introducing a well suited Clément type operator, which requires some technical assumptions that we introduce below and which imposes some constraints on the topology of the meshes. The specific factors will be estimated directly in Lemma 4.8 and Lemma 4.9.

Notations and definitions. For any $K \in \mathcal{T}_h$, we denote by Δ_K, resp. Δ_e, the union of all elements that share at least one vertex, resp. one edge, with K. We denote by $\mathcal{N}_h$, $\mathcal{N}_K$, and N_e, the set of all vertices of elements of $\mathcal{T}_h^j$, $1\le j\le J$, of a given element K, and of a given edge e, respectively. With each vertex z we associate the corresponding unique continuous, piecewise affine function that takes the value 1 at z and vanishes at all other vertices. We denote by $\mathcal{E}^-_{h,\gamma_i}$ the set $\mathcal{E}_{h,\gamma_i}$ (recall that it coincides with $\mathcal{E}_{h,\ell,\gamma_i}$ when ℓ is such that $M_h(\gamma_i) = M_h^{1,\ell}(\gamma_i)$ or $M_h(\gamma_i) = M_h^{1,\ell,*}(\gamma_i)$) and $\mathcal{E}^+_{h,\gamma_i} = \mathcal{E}_{h,3-\ell,\gamma_i}$. We denote also by N_C^- and N_C^+ the set of the vertices which are endpoints of $e \in \mathcal{E}^-_{h,\gamma_i}$ and $\mathcal{E}^+_{h,\gamma_i}$ respectively.

We need the following assumption (see Figure 2), that allows to handle the adaptivity with nonconforming meshes

Assumption 1 . *Each element of $\mathcal{E}^-_{h,\gamma_i}$ is the union of some entire elements of $\mathcal{E}^+_{h,\gamma_i}$ and the number of such elements is independent of h.*

This assumption is easy to satisfy in the practical implementation of the adaptive strategy.

Assumption 2 . *For any two subdomains $\overline{\Omega}_i$ and $\overline{\Omega}_j$ sharing at least one point, there is a connected path passing from $\overline{\Omega}_i$ to $\overline{\Omega}_j$ trough adjacent subdomains such that α is monotone along this path (adjacent means they share a common edge).*

The following proposition is proved in the Appendix

Proposition 4.7. *Under assumption 2, there exists an operator R_h from $\mathbf{V}$ into $\mathbf{V}_h \cap H_0^1(\Omega)^2$, and a constant $C>0$ depending only on the shape parameter of $\mathcal{T}_h$,*

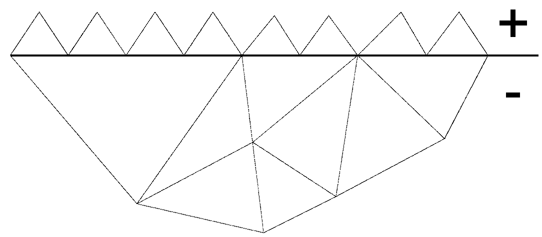

FIGURE 2. Graphic representation of assumption 1

such that: for any $\mathbf{v} \in \mathbf{V}$*, every element* K *and every edge* e *of* K*, the following estimates hold*

$$
\begin{aligned}
\|\mathbf{v} - R_h\mathbf{v}\|_{L^2(K)^2} &\leq C h_K \alpha_K^{-\frac{1}{2}} \|\mathbf{v}\|_{1,\alpha,\Delta_K}, \\
\|\mathbf{v} - R_h\mathbf{v}\|_{L^2(e)^2} &\leq c h_e^{\frac{1}{2}} \alpha_e^{-\frac{1}{2}} \|\mathbf{v}\|_{1,\alpha,\Delta_e}.
\end{aligned} \tag{47}
$$

Lemma 4.8. *The following inequality holds*

$$
\begin{aligned}
\int_e (\beta_e\lambda_h + (\mathbf{n}_e \cdot \alpha\sigma(\mathbf{u}_h)\,\mathbf{n}_e))\;[\mathbf{w}\cdot\mathbf{n}_e]\;d\tau \leq \\
C h_e^{\frac{1}{2}} \alpha_e^{-\frac{1}{2}} \|(\beta_e\lambda_h + (\mathbf{n}_e \cdot \alpha\sigma(\mathbf{u}_h)\,\mathbf{n}_e)\|_{L^2(e)} \sum_{K\in\mathcal{K}_e} \|\mathbf{u} - \mathbf{u}_h^\alpha\|_{1,\alpha,K},
\end{aligned} \tag{48}
$$

where $\mathcal{K}_e$ *denotes the union of elements* K *sharing the edge* e *and* $\beta_e = 0$ *if* $M_h = M_h^{1,\ell,*}$ *and* $= 1$ *otherwise.*

Proof. We have

$$
\int_e (\beta_e\lambda_h + \mathbf{n}_e\cdot\alpha\sigma(\mathbf{u}_h)\,\mathbf{n}_e)\;[\mathbf{w}\cdot\mathbf{n}_e]\;d\tau \leq \|\beta_e\lambda_h + \mathbf{n}_e\cdot\alpha\sigma(\mathbf{u}_h)\,\mathbf{n}_e\|_{H^{-\frac{1}{2}}(e)} \|\,[\mathbf{w}\cdot\mathbf{n}_e]\,\|_{H^{\frac{1}{2}}(e)}.
$$

Using the inverse inequality [18], we get

$$
\begin{aligned}
\int_e (\beta_e\lambda_h + \mathbf{n}_e\cdot\alpha\sigma(\mathbf{u}_h)\,\mathbf{n}_e)\;[\mathbf{w}\cdot\mathbf{n}_e]\;d\tau &\leq C\,h_e^{\frac{1}{2}} \|(\beta_e\lambda_h + \mathbf{n}_e\cdot\alpha\sigma(\mathbf{u}_h)\,\mathbf{n}_e\|_{L^2(e)} \sum_{K\in\mathcal{K}_e} \|\,[\mathbf{w}\cdot\mathbf{n}_e]\,\|_{H^{\frac{1}{2}}(\partial K)} \\
&\leq C\,h_e^{\frac{1}{2}} \alpha_e^{-\frac{1}{2}} \|(\beta_e\lambda_h + \mathbf{n}_e\cdot\alpha\sigma(\mathbf{u}_h)\,\mathbf{n}_e\|_{L^2(e)} \sum_{K\in\mathcal{K}_e} \|\mathbf{u} - \mathbf{u}_h\|_{1,\alpha,K}.
\end{aligned}
$$

□

Lemma 4.9. *We have*

$$
\frac{b(\mu - \mu_h, \mathbf{u}_h)}{\|\mu\|_{-\frac{1}{2},*}} \leq C\Big(\sum_{i=1}^{I} \sum_{e\in\mathcal{E}_{h,\gamma_i}} \eta_e^2\Big)^{\frac{1}{2}}. \tag{49}
$$

Proof. For each $e \in \mathcal{E}_{h,\gamma_i}$, $1 \leq i \leq I$, we have

$$
\int_e (\mu - \mu_h)\,[\mathbf{u}_h\cdot\mathbf{n}_e]\;d\tau \leq \|\mu - \mu_h\|_{H^{-\frac{1}{2}}(e)} \|\,[\mathbf{u}_h\cdot\mathbf{n}_e]\,\|_{H^{\frac{1}{2}}(e)},
$$

and using the inverse inequality [18], we note that

$$
\|\,[\mathbf{u}_h\cdot\mathbf{n}_e]\,\|_{H^{\frac{1}{2}}(e)} \leq h_e^{-\frac{1}{2}} \|\,[\mathbf{u}_h\cdot\mathbf{n}_e]\,\|_{L^2(e)}.
$$

Extending the operator π_h^1, respectively π_h^0, to $H^{-\frac{1}{2}}(\gamma_i)$, we have

$$\|\mu - \mu_h\|_{H^{-\frac{1}{2}}(e)} \le c\|\mu - \mu_h\|_{(H^{-\frac{1}{2}}(\gamma_i))} \le c\|\mu\|_{(H^{-\frac{1}{2}}(\gamma_i))}$$

Summing up for all $e \in \mathcal{E}_{h,\gamma_i}$, $1 \le i \le I$ yields the result. $\square$

Finally, choosing $\mathbf{w}_h = R_h(\mathbf{w}) \in \mathbf{V} \cap H^1(\Omega)^2$ in (46) and (45), using Lemma 4.7, Lemma 4.8 and Lemma 4.9 yield the upper bound on the error.

Theorem 4.10. *If Assumption 1 is satisfied, there exists a constant C independent of h and α, such that if $M_h(\gamma_i) = M_h^{1,*,\ell}(\gamma_i)$,*

$$\|\mathbf{u} - \mathbf{u}_h\|_\alpha + \Big(\frac{\alpha_m}{\alpha_M}\Big)^{\frac{1}{2}} \|\lambda - \lambda_h\|_{-\frac{1}{2},*} \le C\Big(\sum_{K \in \mathcal{T}_h} (\eta_K^2 + \alpha_K^{-1} h_K^2 \|\mathbf{f} - \mathbf{f}_h\|^2_{L^2(\Omega_C)^2}) + \sum_{i=1}^{I} \sum_{e \in \mathcal{E}_{h,\gamma_i}} (\eta_e^2 + \eta_{C,e}^2)\Big)^{\frac{1}{2}}. \tag{50}$$

Corollary 4.11. *If Assumption 1 is satisfied, there exists a constant C independent of h and α, such that if $M_h(\gamma_i) = M_h^{0,\ell}(\gamma_i)$ or $= M_h^{1,\ell}(\gamma_i)$*

$$\|\mathbf{u} - \mathbf{u}_h\|_\alpha + \|\lambda - \lambda_h\|_{-\frac{1}{2},*} \le C\Big(\sum_{K \in \mathcal{T}_h} (\eta_K^2 + \alpha_K^{-1} h_K^2 \|\mathbf{f} - \mathbf{f}_h\|^2_{L^2(\Omega_C)^2}) + \sum_{i=1}^{I} \sum_{e \in \mathcal{E}_{h,\gamma_i}} (\eta_e^2 + \eta_{C,e}^2)\Big)^{\frac{1}{2}}. \tag{51}$$

4.3. **An upper bound for the indicators.** To complete the a posteriori analysis, we have to show the equivalence of the error and the error indicators which means that we should obtain upper bounds to each error indicator in term of the local discretization error. In order to bound the indicators η_K, η_e and $\eta_{C,e}$, we set $E = (e_{\mathbf{u}}, e_\lambda) = (\mathbf{u} - \mathbf{u}_h, \lambda - \lambda_h)$, then we take a test function $\mathbf{w} \in \mathbf{V}$ and compute

$$\begin{aligned} a_\alpha(e_{\mathbf{u}}, \mathbf{w}) + b(e_\lambda, \mathbf{w}) &= \int_{\Omega_C} \alpha\, \sigma(e_{\mathbf{u}}) : \varepsilon(\mathbf{w})\, d\mathbf{x} + \sum_{i=1}^{I} \sum_{e \in \mathcal{E}_{h,\gamma_i}} \int_e e_\lambda\, [\mathbf{w} \cdot \mathbf{n}_e]\; d\tau \\ &= (\mathbf{f}, \mathbf{w}) - a_\alpha(\mathbf{u}_h, \mathbf{w}) - b(\lambda_h, \mathbf{w}) \\ &= \sum_{K \in \mathcal{T}_h} \Big(\int_K (\mathbf{f}_h + \operatorname{div} \alpha\, \sigma(\mathbf{u}_h)) \cdot \mathbf{w}\, d\mathbf{x} + \int_K (\mathbf{f} - \mathbf{f}_h) \cdot \mathbf{w}\, d\mathbf{x} \\ &\quad - \sum_{e \in \mathcal{E}_K} \mathbf{n}_e \cdot (\alpha \sigma(\mathbf{u}_h))\, \mathbf{w}\, d\mathbf{x}\Big) - b(\lambda_h, \mathbf{w}) \\ &= \sum_{K \in \mathcal{T}_h} \Big(\int_K (\mathbf{f}_h + \operatorname{div} \alpha\, \sigma(\mathbf{u}_h)) \cdot \mathbf{w}\, d\mathbf{x} + \int_K (\mathbf{f} - \mathbf{f}_h) \cdot \mathbf{w}\, d\mathbf{x}\Big) \\ &\quad - \frac{1}{2} \sum_{e \in \mathcal{E}_h} \int_e [\mathbf{n}_e \cdot (\alpha \sigma(\mathbf{u}_h)]\; \mathbf{w}\, d\tau - \sum_{i=1}^{I} \sum_{e \in \mathcal{E}_{h,\gamma_i}} \int_e (\alpha \sigma_n(\mathbf{u}_h) + \lambda_h)\, [\mathbf{w} \cdot \mathbf{n}_e]\; d\tau. \end{aligned} \tag{52}$$

We denote by $r_h \mathbf{w} = (r_h^i \mathbf{w})_{1 \le i \le I}$. If $M_h = M_h^{1,\ell,*}(\gamma_i)$, following the same lines than (52), and taking into account $b(\lambda_h, r_h \mathbf{w}) \le 0$ (by the definition of M_h), we have

(53)

$$\begin{aligned}
a_\alpha(e_{\mathbf{u}}, \mathbf{w}) + b(e_\lambda, \mathbf{w}) = \sum_{K\in\mathcal{T}_h} \Big(\int_K (\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\cdot\mathbf{w}\,d\mathbf{x} + \int_K (\mathbf{f}-\mathbf{f}_h)\cdot\mathbf{w}\,d\mathbf{x} \\
- \sum_{e\in\mathcal{E}_K} \mathbf{n}_e\cdot(\alpha\,\sigma(\mathbf{u}_h))\,\mathbf{w}\,d\mathbf{x}\Big) - b(\lambda_h, \mathbf{w}) \\
\geq \sum_{K\in\mathcal{T}_h} \Big(\int_K (\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\cdot\mathbf{w}\,d\mathbf{x} + \int_K (\mathbf{f}-\mathbf{f}_h)\cdot\mathbf{w}\,d\mathbf{x} \\
- \sum_{e\in\mathcal{E}_K} \mathbf{n}_e\cdot(\alpha\sigma(\mathbf{u}_h))\,\mathbf{w}\,d\mathbf{x}\Big) - b(\lambda_h, \mathbf{w} - r_h\mathbf{w}) \\
= \sum_{K\in\mathcal{T}_h} \Big(\int_K (\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\cdot\mathbf{w}\,d\mathbf{x} + \int_K (\mathbf{f}-\mathbf{f}_h)\cdot\mathbf{w}\,d\mathbf{x}\Big) \\
-\frac{1}{2}\sum_{e\in\mathcal{E}_h}\int_e [\mathbf{n}_e\cdot(\alpha\sigma(\mathbf{u}_h)]\ \mathbf{w}\,d\tau - \sum_{i=1}^{I}\sum_{e\in\mathcal{E}_{h,\gamma_i}}\int_e \alpha\,\sigma_n(\mathbf{u}_h)\,[\mathbf{w}\cdot\mathbf{n}_e]\ d\tau \\
-\sum_{i=1}^{I}\sum_{e\in\mathcal{E}_{h,\gamma_i}}\int_e \lambda_h(\big[\mathbf{w}\cdot\mathbf{n}_e - r_h^i(\mathbf{w}\cdot\mathbf{n}_e)\big]\ d\tau
\end{aligned}$$

.

We will obtain the desired estimates by appropriate choices of $\mathbf{w}$. With each $K \in \mathcal{T}_h$, we associate the bubble function ψ_K equal to the product of the three barycentric coordinates on K. For each $e \in \mathcal{E}_\Omega$, we associate the bubble function ψ_e equal to the product of the two barycentric coordinates on e. We introduce a lifting operator defined as follows: On the reference element $\widehat{K}$, we fix a lifting operator $\widehat{P}$ from polynomial traces on $\hat{e}$ on $\widehat{K}$ that vanish at the endpoints of $\hat{e}$, into polynomials on $\widehat{K}$ that vanish on $\partial\widehat{K}\setminus\hat{e}$. A similar operator is obtained on each K by an affine transformation.

Proposition 4.12. *There exists a constant C independent of h and α such that:* $\forall K \in \mathcal{T}_h$

(54) $$\eta_K \leq C\big(\|\mathbf{u}-\mathbf{u}_h\|_{1,\alpha,\mathcal{G}_K} + \big(\sum_{K'\in\mathcal{G}_K} \alpha_{K'}^{-1} h_{K'}^2 \|\mathbf{f}-\mathbf{f}_h\|^2_{L^2(K')^2}\big)^{\frac{1}{2}}\big),$$

where $\mathcal{G}_K$ is the union of K and all triangles containing an edge of K.

Proof. We will bound each term of η_K which, as standard in a posteriori analysis ([48]), relies on appropriate choices of the test function $\mathbf{w}$ in (52)

(1) We take $\mathbf{w}$ in (52) equal to

$$\mathbf{w} = \begin{cases} (\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\psi_K & \text{in } K, \\ 0 & \text{elsewhere.} \end{cases}$$

Since ψ_K vanishes on ∂K, this yields

$$\|(\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\psi_K^{\frac{1}{2}}\|^2_{L^2(K)^2} \leq$$
$$\int_K \alpha\,\sigma(e_{\mathbf{u}}) : \varepsilon((\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\psi_K)\,d\mathbf{x} - \int_K (\mathbf{f}-\mathbf{f}_h)\cdot(\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\psi_K)\,d\mathbf{x}.$$

It follows that

$$\|(\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\psi_K^{\frac{1}{2}}\|^2_{L^2(K)^2} \le C(\|\mathbf{u}-\mathbf{u}_h\|_{1,\alpha,K}|(\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\psi_K^{\frac{1}{2}}|_{1,\alpha,K} + \|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^2}\|(\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h))\psi_K^{\frac{1}{2}}\|_{L^2(K)^2}.$$

It can be checked by going to the reference element that for any polynomial φ of degree at most k, the following inequalities hold [18] :

$$\|\varphi\|_{L^2(K)} \le C\|\varphi\psi_K^{\frac{1}{2}}\|_{L^2(K)}, \quad |\varphi\psi_K|_{H^1(K)} \le Ch_K^{-1}\|\varphi\|_{L^2(K)},$$

with constants depending only on k and the shape parameter of K. Noting that ψ_K is ≤ 1 (and with obvious extension to vector valued functions), we obtain

$$\alpha_K^{-\frac{1}{2}}h_K\|\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h)\|_{L^2(K)^2} \le c(\|\mathbf{u}-\mathbf{u}_h\|_{1,\alpha,K} + \alpha_K^{-\frac{1}{2}}h_K\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^2}). \tag{55}$$

(2) We denote by e an edge in $\mathcal{E}_K$. We distinguish between two cases.

- First, if e is not contained in $\cup_{i=}^I\gamma_i$, it is a common edge of the two adjacent elements K and K'. Now, we choose $\mathbf{w}$ in (52) to be equal to

$$\mathbf{w} = \begin{cases} P_{K,e}([\alpha\,\sigma(\mathbf{u}_h)\mathbf{n}_e]\,\psi_e) & \text{in } K,\\ P_{K',e}([\alpha\,\sigma(\mathbf{u}_h)\mathbf{n}_e\cdot]\,\psi_e) & \text{in } K',\\ 0 & \text{elsewhere,}\end{cases}$$

This yields

$$\|\,[\alpha\sigma(\mathbf{u}_h)\mathbf{n}_e]\;\psi_e^{\frac{1}{2}}\|^2_{L^2(e)^2} \le \sum_{\kappa\in(K,K')}|\mathbf{u}-\mathbf{u}_h|_{1,\alpha,\kappa}|P_{\kappa,e}([\alpha\sigma(\mathbf{u}_h)\mathbf{n}_e]\,\psi_e)|_{1,\alpha,\kappa}$$
$$+ (\|\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h)\|_{L^2(\kappa)^2} + \|\mathbf{f}-\mathbf{f}_h\|_{L^2(\kappa)^2})\|P_{\kappa,e}([\alpha\sigma(\mathbf{u}_h)\mathbf{n}_e]\,\psi_e)\|_{L^2(\kappa)^2}.$$

The following inequalities, obtained by going to the reference element, hold

$$\|\,[\alpha\,\sigma(\mathbf{u}_h)\,\mathbf{n}_e]\,\|_{L^2(e)^2} \le c\|\,[\alpha\,\sigma(\mathbf{u}_h)\mathbf{n}_e]\,\psi_e^{\frac{1}{2}}\|_{L^2(e)^2},$$
$$|P_{\kappa,e}([\alpha\,\sigma(\mathbf{u}_h)\mathbf{n}_e]\,\psi_e)|_{H^1(\kappa)^2} + h_e^{-1}\|P_{\kappa,e}([\alpha\,\sigma(\mathbf{u}_h)\mathbf{n}_e]\;\psi_e)\|_{L^2(\kappa)^2} \le C\,h_e^{-\frac{1}{2}}\|\,[\alpha\,\sigma(\mathbf{u}_h)\,\mathbf{n}_e]\,\|_{L^2(e)^2}.$$

Noting that $ch_\kappa \le h_e \le h_\kappa$, we obtain

$$\alpha^{-\frac{1}{2}}h_e^{\frac{1}{2}}\|\,[\alpha\,\sigma(\mathbf{u}_h)\,\mathbf{n}_e]\,\|_{L^2(e)^2} \le C\sum_{\kappa\in(K,K')}(|\mathbf{u}-\mathbf{u}_h|_{1,\alpha,\kappa} + \alpha_K^{-\frac{1}{2}}h_\kappa\|\mathbf{f}-\mathbf{f}_h\|_{L^2(\kappa)^2} + \alpha_K^{-\frac{1}{2}}h_\kappa\|\mathbf{f}_h + \operatorname{div}\alpha\,\sigma(\mathbf{u}_h)\|_{L^2(\kappa)^2}). \tag{56}$$

- Let $e \in \cup_{i=1}^I\gamma_i$. If $e \in \mathcal{E}_{\gamma_i}^+$, for a given i, we denote by K' the element such that e is contained in $e' \in \mathcal{E}_{K'}$. We extend $[\alpha\sigma(\mathbf{u}_h)\mathbf{n}_e\,]\,\psi_e$ to the entire e' by zero and we make the same choice of $\mathbf{w}$ as previously and we obtain (56). If $e \in \mathcal{E}_{\gamma_i}^-$, then according to assumption 1, we replace K' by the finite number of elements K_i, which share the edge e, we define $\mathbf{w}$ as previously with respect to this change and we proceed similarly to the previous case to obtain the estimate (56)

□

Corollary 4.13. *If Assumption 1 is satisfied, there exists a constant C independent of h and α such that*

$$(57)\qquad \Big(\sum_{K\in\mathcal{T}_h}\eta_K^2\Big)^{\frac12}\le C\,\Big(\|\mathbf{u}-\mathbf{u}_h\|_\alpha+\Big(\sum_{K\in\mathcal{T}_h}\alpha_K^{-1}h_K^2\|\mathbf{f}-\mathbf{f}_h\|^2_{L^2(K)^2}\Big)^{\frac12}\Big),$$

Proposition 4.14. *If Assumption 1 is satisfied, we take $\beta_e=0$ if $M_h(\gamma_i)=M_h^{1,\ell,*}(\gamma_i)$ and $\beta_e=1$ otherwise, then, there exists a constant C independent of h and α such that*

$$(58)\qquad \eta_{C,e}\le C\big(\|\mathbf{u}-\mathbf{u}_h\|_{1,\alpha,\mathcal{G}_e}+\alpha_e^{-\frac12}h_e^{\frac12}\|\lambda-\lambda_h\|_{L^2(e)} + (1-\beta_e)\,\alpha_e^{-\frac12}h_e^{\frac54}\|\lambda_h\|_{L^2(e)}+\Big(\sum_{\kappa\in\mathcal{G}_e}\alpha_\kappa^{-1}h_\kappa^2\|\mathbf{f}-\mathbf{f}_h\|^2_{L^2(\kappa)^2}\Big)^{\frac12}\big),$$

where $\mathcal{G}_e$ is the union of all triangles having a non null intersection with e.

Remark 4.15. *Note that either $\beta_e=1$ and the third term in the right-hand side vanishes, or $\beta_e=0$ and it is of high order. In all cases it could be neglected.*

Proof. Let us denote by $\sigma_n(\mathbf{u})=(\mathbf{n}\cdot\sigma(\mathbf{u}))\mathbf{n}$. For $e\in\mathcal{E}_{h,\gamma_i}$, $1\le i\le I$, we take $\mathbf{w}$ in (52) equal to

$$\mathbf{w}=\begin{cases} P_{K,e}(\psi_e(\beta_e\lambda_h\,\mathbf{n}_e+\alpha\,\sigma(\mathbf{u}_h)\mathbf{n}_e)) & \text{in } K,\\ 0 & \text{elsewhere.}\end{cases}$$

with $\beta_e=1$ if $M_h(\gamma_i)=M_h^0(\gamma_i)$ or $M_h^{1,\ell}(\gamma_i)$ and $\beta_e=0$ $M_h(\gamma_i)=M_h^{1,\ell,*}(\gamma_i)$.

- If $M_h(\gamma_i)=M_h^0(\gamma_i)$ or $M_h^{1,\ell}(\gamma_i)$ then arguing as in the previous proposition, we deduce that

$$\|(\lambda_h+\alpha\sigma_n(\mathbf{u}_h)\,\psi_e^{\frac12}\|^2_{L^2(e)^2}\le\|\mathbf{u}-\mathbf{u}_h\|_{1,\alpha,K}|P_{\kappa,e}([\lambda_h\mathbf{n}_e+\alpha\sigma(\mathbf{u}_h)\mathbf{n}_e]\,\psi_e)|_{1,\alpha,K}+ (\|\mathbf{f}_h+\operatorname{div}\alpha\,\sigma(\mathbf{u}_h)\|_{L^2(K)^2}+\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^2})\|P_{\kappa,e}([\lambda_h\mathbf{n}_e+\alpha\sigma(\mathbf{u}_h)\mathbf{n}_e]\;\psi_e)\|_{L^2(K)^2} +\int_e(\lambda-\lambda_h)\,(\lambda_h+\alpha\sigma_n(\mathbf{u}_h))\,\psi_e\,d\tau).$$

  Thus, we obtain

$$(59)\qquad \alpha_e^{-\frac12}h_e^{\frac12}\|\lambda_h+\alpha\sigma_n(\mathbf{u}_h)\|_{L^2(e)}\le C(\|\mathbf{u}-\mathbf{u}_h|_{1,\alpha,K}+\alpha_K^{-\frac12}h_K\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^2}+ \alpha_K^{-\frac12}h_K\|\mathbf{f}_h+\operatorname{div}\alpha\sigma(\mathbf{u}_h)\|_{L^2(K)^2})+\alpha_e^{-\frac12}h_e^{\frac12}\|\lambda-\lambda_h\|_{L^2(e)}.$$

- If $M_h(\gamma_i)=M_h^{1,\ell,*}(\gamma_i)$: using (52), we deduce as in the previous case

$$\|\alpha\sigma_n(\mathbf{u}_h)\,\psi_e^{\frac12}\|^2_{L^2(e)^2}\le\|\mathbf{u}-\mathbf{u}_h\|_{1,\alpha,K}|P_{\kappa,e}([\alpha\sigma_n(\mathbf{u}_h)]\;\psi_e)|_{1,\alpha,K}+ (\|\mathbf{f}_h+\operatorname{div}\alpha\sigma(\mathbf{u}_h)\|_{L^2(K)^2}+\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^2})\|P_{\kappa,e}([\alpha\sigma_n(\mathbf{u}_h)]\;\psi_e)\|_{L^2(K)^2} +\int_e(\lambda-\lambda_h)\,(\alpha\sigma_n(\mathbf{u}_h))\,\psi_e\,d\tau)+\int_e\lambda_h\,(\alpha\sigma_n(\mathbf{u}_h)-r_h^i(\alpha\sigma_n(\mathbf{u}_h)))\,\psi_e\,d\tau).$$

  The last term and (19) yield the additional extra term $\alpha_e^{-\frac12}h_e^{\frac54}\|\lambda_h\|_{L^2(e)}$ on the right-hand side of (59).

□

Corollary 4.16. *If Assumption 1 is satisfied, there exists a constant C independent of h and α such that,*

$$(\sum_{i=1}^{I} \sum_{e \in \mathcal{E}_{h,\gamma_i}} \eta_{C,e}^2)^{\frac{1}{2}} \leq C\,(\|\mathbf{u} - \mathbf{u}_h\|_\alpha + (\sum_{i=1}^{I} \sum_{e \in \mathcal{E}_{h,\gamma_i}} \alpha_e^{-1}\, h_e \|\lambda - \lambda_h\|_{L^2(e)}^2)^{\frac{1}{2}} + (\sum_{K \in \mathcal{T}_h} \alpha_K^{-1}\, h_K^2 \|\mathbf{f} - \mathbf{f}_h\|_{L^2(K)^2}^2)^{\frac{1}{2}}). \tag{60}$$

Remark 4.17. *The second term in the right-hand side is bounded by a constant times the norm $H^{-\frac{1}{2}}(\Gamma_C)$. However, the constant depends on the ratio $(\frac{\alpha_m}{\alpha_M})$. Considering the a posteriori estimate* (60) *(with regard to the error for the Lagrange multipliers), shows that when the ratio between adjacent bodies (sharing the same contact zone) is large, nonconforming approximation is more suited.*

It now remains to bound η_e.

Proposition 4.18. *If Assumption 1 is satisfied, there exists a constant C independent of h and α such that the following estimate holds*

$$\eta_e \leq C\,\gamma_e\,\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\mathcal{G}_e)^2} \tag{61}$$

where $\mathcal{G}_e$ denotes, as before, the union of all elements having a non null intersection with e and $\gamma_e = 1 + \max(\alpha_K^{-1}, \alpha_{K'}^{-1})$.

Proof. We define a function μ_h on $\cup_{i=1}^{I} \mathcal{E}_{h,\gamma_i}$ by

$$\mu_h = \begin{cases} [\mathbf{u}_h \cdot \mathbf{n}_e]\ \psi_e & \text{on } e, \\ 0, & \text{elsewhere.} \end{cases}$$

$$b(\mu_h, \mathbf{u}_h) = b(\mu_h, \mathbf{u}_h - \mathbf{u}) \tag{62}$$

- If $\mathcal{E}_{h,\gamma_i} = \mathcal{E}_{h,\gamma_i}^{+}$ then we consider the two elements K, K' from both sides of γ_i such that e is an entire edge of K and is contained on an entire edge e' of K'. We extend the product $\mu_h \psi_e$ by zero to e' and then we solve the problem: for $\kappa \in (K, K')$, find $\varphi \in H^1_{0,\partial\kappa\setminus e}(\kappa)$

$$\begin{cases} \Delta\varphi = 0, & \text{in } \kappa, \\ \varphi = \mu_h, & \text{on } e, \\ \varphi = 0, & \text{elsewhere,} \end{cases} \tag{63}$$

where $H^1_{0,\partial\kappa\setminus e}(\kappa) = \{v \in H^1(\kappa);\ v = 0 \text{ on } \partial\kappa \setminus e\}$. From the definition of μ_h, $b(.,.)$, (62) and integration by parts, it follows that

$$\begin{aligned} \|\,[\mathbf{u}_h \cdot \mathbf{n}_e]\,\psi_e^{\frac{1}{2}}\,\|_{L^2(e)}^2 &= b(\mu_h, \mathbf{u}_h) \\ &= b(\mu_h, \mathbf{u}_h - \mathbf{u}) \\ &= \int_e \mu_h\ [(\mathbf{u}_h - \mathbf{u}) \cdot \mathbf{n}_e]\ d\tau \quad = \sum_{\kappa \in (K,K')} (\int_\kappa \mathbf{grad}\,\varphi \cdot (\mathbf{u}_h - \mathbf{u})\,d\mathbf{x} \\ &- \int_\kappa \varphi\,\mathrm{div}\,(\mathbf{u}_h - \mathbf{u})\,d\mathbf{x}) \end{aligned}$$

and we deduce that

$$\begin{aligned}&\|\,[\mathbf{u}_h\cdot\mathbf{n}_e]\,\|^2_{L^2(e)}\le\|\,[\mathbf{u}_h\cdot\mathbf{n}_e]\,\psi_e^{\frac12}\|^2_{L^2(e)}\\&\quad\le C\sum_{\kappa\in(K,K')}(|\varphi|_{H^1(\kappa)}\|\mathbf{u}_h-\mathbf{u}\|_{L^2(\kappa)^2}+h_\kappa^{-1}\|\varphi\|_{L^2(\kappa)}\,h_\kappa\,\|\mathrm{div}\,(\mathbf{u}_h-\mathbf{u})\|_{L^2(\kappa)}|.\end{aligned}$$

By going to the reference element and using direct estimates on the boundary value problem (63), we have

$$\begin{aligned}h_\kappa^{-1}\|\varphi\|_{L^2(\kappa)}+|\varphi|_{H^1(\kappa)}&\le C\,\|\,[\mathbf{u}_h\cdot\mathbf{n}_e]\,\psi_e^{\frac12}\|_{L^2(e)}\\&\le C\,h_e^{-\frac12}\|\,[\mathbf{u}_h\cdot\mathbf{n}_e]\,\|_{L^2(e)}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|\,[\mathbf{u}_h\cdot\mathbf{n}_e]\,\|^2_{L^2(e)}\le C\,h_e^{-\frac12}\|\,[\mathbf{u}_h\cdot\mathbf{n}_e]\,\|_{L^2(e)}\big(&\sum_{\kappa\in(K,K')}(\|\mathbf{u}_h-\mathbf{u}\|_{L^2(\kappa)^2}+\\&h_\kappa\,\|\mathrm{div}\,(\mathbf{u}_h-\mathbf{u})\|_{L^2(\kappa)})\big)+\|\,[\mathbf{u}_h\cdot\mathbf{n}_e]\,\|_{L^2(e)}\end{aligned}$$

Noting that $ch_\kappa\le h_e\le c'h_\kappa$ we have

$$\begin{aligned}\|\,[\mathbf{u}_h\cdot\mathbf{n}_e]\,\|_{L^2(e)}&\le Ch_e^{-\frac12}\sum_{\kappa\in(K,K')}h_\kappa\|\mathbf{u}_h-\mathbf{u}\|_{H^1(\kappa)^2}\\&\le Ch_e^{\frac12}\sum_{\kappa\in(K,K')}\|\mathbf{u}_h-\mathbf{u}\|_{H^1(\kappa)^2},\end{aligned}$$

thus,

$$h_e^{-\frac12}\,\|\,[\mathbf{u}_h\cdot\mathbf{n}_e]\,\|_{L^2(e)}\le C\,\delta_e\sum_{\kappa\in(K,K')}\|\mathbf{u}_h-\mathbf{u}\|_{1,\alpha,\kappa},\tag{64}$$

where $\delta_e=1+\max(\alpha_K^{-1},\alpha_{K'}^{-1})$. which is the disred estimate.

- If $\mathcal{E}_{h,\gamma_i}=\mathcal{E}^-_{h,\gamma_i}$, then it follows from Assumption 1, that e is an entire edge of an element K on one side and is a union of edges e'_i, $i=1,\ldots,i^*$. Since $\mathbf{u}_h$ belongs to $\mathbf{V}_h$ it is continuous at the endpoints of e'_i, therefore we can still solve (63) with K' replaced by $\Delta_e\cap\Omega^+_{\mathcal{C}}$. Arguing as in the previous case we obtain the desired estimate.

□

Corollary 4.19. *If Assumption 1 is satisfied, there exists a constant C independent of h and α such that*

$$(\sum_{i=1}^{I}\sum_{e\in\mathcal{E}_{h,\gamma_i}}\eta_e^2)^{\frac12}\le C\,\|\mathbf{u}-\mathbf{u}_h\|_\alpha.\tag{65}$$

5. Numerical experiments

First, we describe briefly the implementation of discrete problem (17). All of the numerical experiments are achieved with FreeFem++ [31]. The discrete solution $(\mathbf{u}_h,\lambda_h)$ is a saddle-point of the Lagrangian functional defined over $\mathcal{K}_h$ by

$$\mathcal{L}(\mathbf{v}_h,\mu_h)=\frac12 a_\alpha(\mathbf{v}_h,\mathbf{v}_h)-L(\mathbf{v}_h)-b(\mu_h,\mathbf{v}_h),$$

which means that it satisfies the min-max principle

$$\mathcal{L}(\mathbf{u}_h,\mu_h) \le \mathcal{L}(\mathbf{u}_h,\lambda_h) \le \mathcal{L}(\mathbf{v}_h,\lambda_h), \quad \forall(\mathbf{v}_h,\mu_h) \in \mathcal{K}_h. \tag{66}$$

Let $\mathbf{V}$, $\mathbf{U}_\alpha$ denote the vectors with the entries given by the nodal values of the functions $(\mathbf{v}_h,\mu_h)$ and $(\mathbf{u}_h,\lambda_h)$, respectively. Let $\mathbf{M}$ and Λ be the vectors with the entries given by the nodal values of μ_h and λ_h, respectively, for the three different choices of the space M_h, namely $M_h(\gamma_i) = M_h^1(\gamma_i)$, $M_h(\gamma_i) = M_h^{1,*}(\gamma_i)$ or $M_h(\gamma_i) = M_h^0(\gamma_i)$, $1 \le i \le I$. The saddle-point problem for the Lagrangian (66) can be rewritten in the finite dimensional setting :
Find $\mathbf{U}_\alpha = (\mathbf{u}_h,\lambda_h)$ and Λ, solution of the following max-min problem

$$\max_{PM\ge 0}\Big(\min_{\mathbf{V}} \frac{1}{2}{}^t\mathbf{V}\mathbf{K}\mathbf{V} - {}^t\mathbf{V}\mathbf{F} + ({}^t\mathbf{V}\,\mathbf{L})PM\Big), \tag{67}$$

where the matrix $\mathbf{K}$ denotes the stiffness matrix, $\mathbf{L}$ the coupling matrix and P expresses the sign conditions for the multipliers. $\mathbf{F}$ denotes the vector corresponding to the external loads.

Given the triangulations $\mathcal{T}_h^j$ of Ω_j, $1 \le j \le J$, let N_j, $1 \le j \le J$ denote the number of nodes in Ω_j. We introduce the finite element basis of $\mathbf{V}_h(\Omega_j)$:

$$(\eta_1,\dots,\eta_{2N_j}) = ((w_1\mathbf{e}_1, w_1\mathbf{e}_2),\dots,(w_{N_j}\mathbf{e}_1, w_{N_j}\mathbf{e}_2)), \quad j = 1,\dots,J,$$

where $(w_i)_i$ denotes the (scalar) affine Lagrange finite element basis and $(\mathbf{e}_1,\mathbf{e}_2)$ the canonical basis in $\mathbb{R}^2$. Then the matrix $\mathbf{K}$ is defined by

$$\mathbf{K} = \begin{pmatrix} K^1 & 0 & \dots \\ 0 & K^2 & 0\dots \\ & \dots & \\ & \dots & K_J \end{pmatrix},$$

$$(K^j)_{ns} = a_\alpha(\eta_n,\eta_s) = \int_{\Omega_j} \varepsilon(\eta_n) : A^\alpha \varepsilon(\eta_s)\, d\mathbf{x}, \quad n,s = 1,\dots,2N_j,$$

and the right-hand side takes the form $\mathbf{F} = (F^j)$, with $F^j = (\mathbf{f}_n^j)_n$, $D = (\int_{\Omega_j} \eta_n\,\eta_s\,dx)_{ns}$, $n,s = 1,\dots,2N_j$.

Since each contact zone γ_i occurs between two bodies $\Omega_{j1(i)}$ and $\Omega_{j2(i)}$, we will drop the index i, and we denote by Ω_1 and Ω_2, the subdomains in contact at γ_i. Let m_ℓ denote the number of nodes of Ω_ℓ on γ. We fix $\ell = 1$ as a Lagrange multiplier side and we define $(\psi_k)_k$, $1 \le k \le m_1 - 1$ to be the finite element basis associated with $W_h^{1,1}(\gamma)$ and $(\varphi_k)_k$, $1 \le k \le m_1 - 1$, to be the finite element basis associated with $W_h^{0,1}(\gamma)$.

If $M_h(\gamma)$ is $= M_h^{0,1}(\gamma)$ or $= M_h^{1,1}(\gamma)$, then P is given by the identity matrix, else $M_h(\gamma) = M_h^{1,1,*}(\gamma)$, and $P_{ij} = \int_\gamma \psi_i\,\psi_j\,d\tau$, $1 \le i,j \le m_1 - 1$.

Finally, the *coupling* matrix $\mathbf{L} = \begin{pmatrix} L^1 \\ L^2 \end{pmatrix}$ is defined in the following way

- If $M_h(\gamma) = M_h^{1,1}(\gamma)$ or $M_h(\gamma) = M_h^{1,1,*}(\gamma)$, then
$$(L^\ell)_{ij} = \int_\gamma \psi_j\,(\eta_i \cdot \mathbf{n}^\ell)\,d\tau,\ 1 \le i \le N_\ell,\ 1 \le j \le m_1 - 1.$$
- If $M_h(\gamma) = M_h^{0,1}(\gamma)$, then
$$(L^\ell)_{ij} = \int_\gamma \varphi_j\,(\eta_i \cdot \mathbf{n}^\ell)\,d\tau,\ 1 \le i \le N_\ell,\ 1 \le j \le m_1 - 1.$$

In order to solve the discrete problem we will use an adapted semi-smooth Newton method [34, 36] which consists in finding the contact zone

$$\mathcal{S} = \left\{ \lambda_h \in W_h^{k,1}(\gamma),\ k = 0, 1 \ \ P\lambda_h < 0 \right\}. \tag{68}$$

We use the algorithm of primal-dual active set strategy [[36], algorithm (2.9)]. The stopping criterion consists in checking if the residual vanishes at the endpoints of each connected component of a stable set $\mathcal{S}$. In practice the algorithm is very fast and very efficient.

Remark 5.1. *One has to pay attention to the initial guess in the algorithm which has to contain the full reached contact zone. In practice, such a choice is always possible because the candidate area to the contact is always known. If the initial guess does not fulfill this condition, it may occur that the missed part of the contact zone is not obtained when convergence is reached.*

5.1. **Adaptive strategy.** We start always with a fixed uniform or quasi-uniform triangulation $\mathcal{T}_{h,n}$, $n = 1$. Next, we perform iteratively the following adaptivity step:

On the triangulation $\mathcal{T}_{h,n}$, we compute the solution $(\mathbf{u}_h, \lambda_h)$ of problem (17), the corresponding error indicators as defined in (35), (36) and (37) and the mean value

$$\overline{\eta}_h^n = \frac{1}{N_h^n} \sum_{K \in \mathcal{T}_{h,n}} \eta_K, \tag{69}$$

where N_h^n is the number of triangles in $\mathcal{T}_{h,n}$. Next, each triangle K such that $\eta_K \geq \overline{\eta}_h^n$, is divided in such a way so that the diameters of the new triangles inside it are very close to h_K times the ratio $\dfrac{\overline{\eta}_h^n}{\eta_K}$.

When $\eta_e + \eta_C$ is of the same order as η_K we perform a second step of the adaptivity on the edges of Γ_C following the same strategy as above.

The adaptivity is performed either a fixed (small) number of times or until the quantity $\overline{\eta}_h^n$ becomes smaller than a given tolerance.

5.2. **Example 1.** In this case the data are as follows: we consider the L-shaped domain $\Omega_1 =]-10, 0[\times]0, 1[\cup]-1, 0[\times]-3, 1[$, $\Omega_2 =]-15, -5[\times]-2, 0[$, $\Gamma_c =]-10, -5[\times\{0\}$, $\nu = 0.3$, $E = 215$. The surface force is $f(x) = -1$ and is defined on the boundary $\Gamma_N =]-1, 0[\times\{-3\}$. Also,

$$\alpha(x, y) = \begin{cases} 1 & \text{in}\,\Omega_1, \\ 10^{-3} & \text{in}\,\Omega_2, \end{cases}$$

We show the initial and the final deformed mesh after 10 cycles of mesh refinement, as well as the error indicators on Figures 3-4 and on Figures 5-6

5.3. **Example 2.** The T-shaped domain is a union of two rectangles $\Omega_1 =]-5, 5[\times]0, 1[\cup]-1, 1[\times]-4, 1[$ and the support part is $\Omega_2 =]-8, -1.5[\times]-2, 0[$ and $\Omega_3 =]1.5, 8[\times]-2, 0[$, $\Gamma_c =]-5, -1.5[\times\{0\}\cup]1.5, 5[\times\{0\}$, $\nu = 0.3$, $E = 2150$. The gravity force is $g = -0.1$. To remove free sliding in the x-direction we impose zero displacement on $\{0\}\times]-5, 1[$. We give the initial deformed mesh in Figure 7 and the final one in Figure 8 for $\alpha = 1$ in the entire domain.

We give the same plots in Figures 9-10 for the case

$$\alpha(x, y) = \begin{cases} 1 & \text{in}\,\Omega_1, \\ 10^{-2} & \text{in}\,\Omega_2 \cup \Omega_3. \end{cases}$$

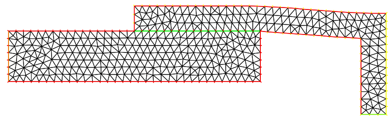

Figure 3. Initial mesh

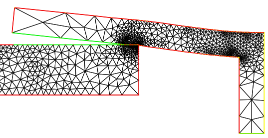

Figure 4. Final mesh after 10 cycle of adaptivity

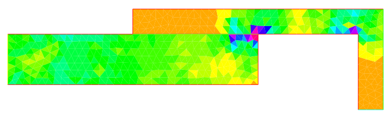

Figure 5. Initial error indicator

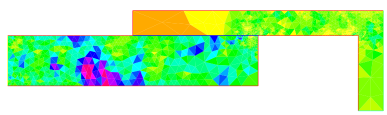

Figure 6. Final error indicator

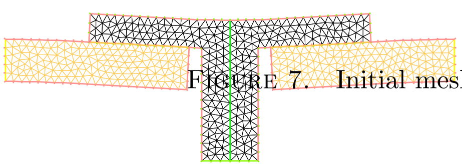

Figure 7. Initial mesh

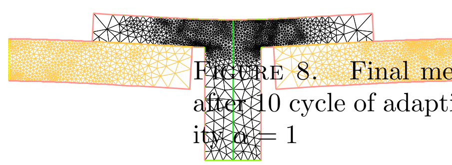

Figure 8. Final mesh after 10 cycle of adaptivity $\alpha = 1$

The convergence of the adaptive strategy is given on Figure 11. We have computed a reference solution on a very fine uniform mesh with a constant mesh size of 0.05. The mesh size in the coarse initial mesh is 0.3 in all examples. We plot, as a function of iteration numbers, the global error indicator η, the error in the energy norm and the error indicator η-mortar, the Hilbertian sum of the error indicators on the contact zone. We observe in this example that both η and η-mortar decrease

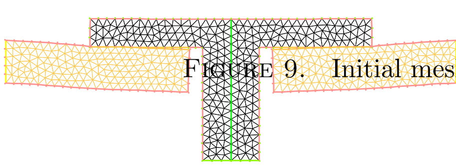

Figure 9. Initial mesh

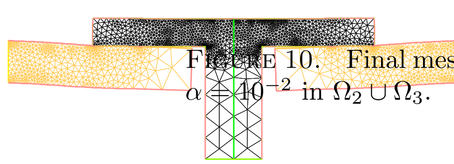

Figure 10. Final mesh $\alpha = 10^{-2}$ in $\Omega_2 \cup \Omega_3$.

with the energy norm of the error. We also remark that the η-mortar is negligible for the adaptive process in this example and this is not surprising since solutions of the Signorini unilateral contact problems with straight contact zones are smooth, see[43] for more details.

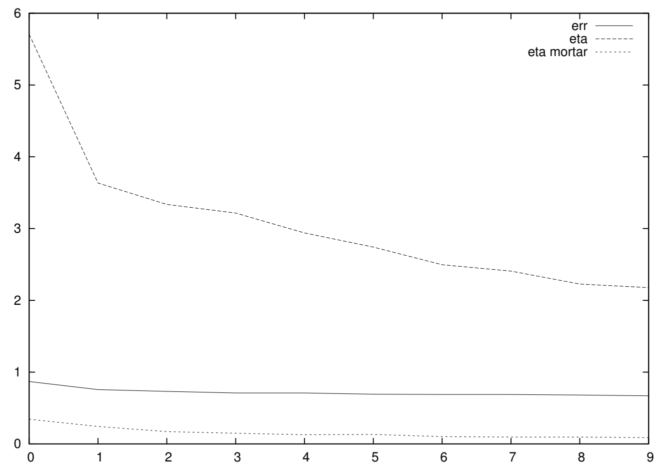

FIGURE 11. Convergence of the adaptivity

Finally, we perform another experiment with

$$\alpha(x,y) = \begin{cases} 1 & \text{in } \Omega_1, \\ 10^{-1} & \text{in } \Omega_2, \\ 10^{-2} & \text{in } \Omega_3. \end{cases}$$

We plot the final deformed mesh in Figure 12 where we can see the influence of the parameter α in the adaptive strategy

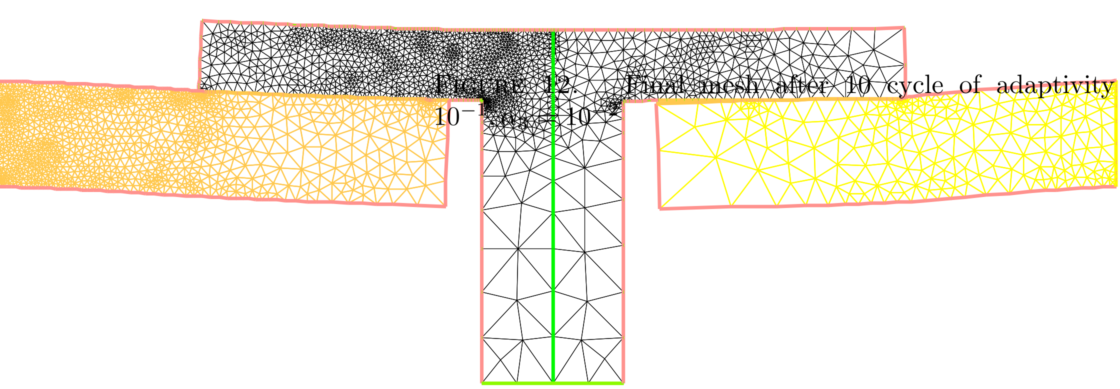

FIGURE 12. Final mesh after 10 cycle of adaptivity $\alpha_2 = 10^{-1}, \alpha_3 = 10^{-2}$

Appendix

In this appendix, we give the proof of proposition 4.7. The proof consists in constructing a regularized interpolation operator with the appropriate approximation properties. Such operators are based on local modified quasi-interpolation operators of Clément type [10]. In our case the construction has to take into account the nonhomogeneous materials in the spirit of [12], the non-compatibility of the meshes at the cracks $\cup_{i=1}^{I}\gamma_i$, and the new (and major) difficulty which is the discontinuity (even) of the solution of the continuous problem at the contact zone $\cup_{i=1}^{I}\gamma_i$.

Given $z \in (\mathcal{N}_h \setminus \mathcal{N}_C^+) \cup \mathcal{N}_C^-$, let ω_z denote the support of the nodal basis function φ_z. It is the union of all elements that have z as a vertex. With each z we associate $\ell(z)$ in $\{1, \dots, J\}$, such that

- z belongs to $\overline{\Omega}_{\ell(z)}$.
- $\alpha_{\ell(z)}$ is maximal among α_j, $j = 1, \dots, J$ such that $\overline{\Omega}_j$ contain z.

We denote by

$$\oint_\omega v\, d\mathbf{x} = \frac{1}{\mathrm{meas}(\omega)} \int_\omega v\, d\mathbf{x}$$

the mean value of the function v on the set ω. Then, we set

$$\pi_z v = \begin{cases} \oint_{\omega_z \cap \Omega_{\ell(z)}} v\, d\mathbf{x}, & \text{if } z \in (\Omega \setminus \mathcal{N}_C^+) \cup \mathcal{N}_C^-, \\ 0 & \text{if } z \in \partial\Omega \cup \mathcal{N}_C^+. \end{cases} \tag{70}$$

We define the quasi-interpolation operators $R_h^1 : L^2(\Omega)^2 \mapsto \mathbf{V}_h$

$$R_h^1 v = \sum_{z \in \mathcal{N}_h \setminus \mathcal{N}_C^+} (\pi_z v_i)\varphi_z, \tag{71}$$

with the obvious notation $R_h^1 \mathbf{v} = (R_h^1 v_1, R_h^1 v_2)$.

In order to enforce the continuity at the vertices of $\mathcal{N}_C^+$, we define the affine piecewise continuous function on γ

$$\Phi(z) = (\varphi_1, \varphi_2)(z) = R_h^1(z), \quad \forall z \in N_C^-.$$

We define

$$R_h \mathbf{v} = \begin{cases} \Phi(z) & \text{if } z \in N_C^+ \setminus \mathcal{N}_h, \\ R_h^1 & \text{otherwise.} \end{cases} \tag{72}$$

It is clear that $R_h : L^2(\Omega)^2 \mapsto \mathbf{V}_h \cap H_0^1(\Omega)^2$. For arbitrary $K \in \mathcal{T}_h$ and for each component of $\mathbf{v} = (v_1, v_2) \in \mathbf{V}$, we have

$$\|v_i - R_h v_i\|_{L^2(K)} = \|\sum_{z \in \mathcal{N}_K} \varphi_z (v_i - R_h v_i)\|_{L^2(K)} \le \sum_{z \in \mathcal{N}_K} \|\varphi_z (v_i - R_h v_i)\|_{L^2(K)}.$$

We have to distinguish between several cases.

- We consider a vertex z which is not contained in the boundary of any of two subdomains. Then from the Bramble-Hilbert inequality, we deduce

  $$\begin{aligned} \|\varphi_z (v_i - R_h v_i)\|_{L^2(K)} &\le \|v_i - R_h v_i\|_{L^2(K)} \le \|v_i - R_h v_i\|_{L^2(\omega_z)} \\ &\le c\, \mathrm{diam}\, \omega_z |v_i|_{H^1(\omega_z)} \le c\, h_K \alpha_K^{-\frac{1}{2}} \|v_i\|_{1,\alpha,\Delta_K}. \end{aligned}$$

  When $z \in \partial\Omega$, similar computations with the Poincaré-Friedrichs inequality lead to the same estimate.

- Consider now z which is not in $\partial\Omega$ but is in $\partial\Omega_{\ell(K)}$ where $\ell(K)$ is such that $K \in \overline{\Omega_{\ell(K)}}$. If $\ell(K) = \ell(z)$, then the previous argument with ω_z replaced by $\omega_z \cap \Omega_{\ell(K)}$ still applies.
- If $\ell(K) \neq \ell(z)$, we have

$$\begin{aligned}\|\varphi_z(v_i - R_h v_i)\|_{L^2(K)} &\leq \|\varphi_z(v_i - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i\, d\mathbf{x})\|_{L^2(K)} \\ &\leq \|\varphi_z(v_i - \oint_{\omega_z\cap\Omega_{\ell(K)}} v_i\, d\mathbf{x})\|_{L^2(K)} \\ &+ \|\varphi_z(\oint_{\omega_z\cap\Omega_{\ell(K)}} v_i - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i\, d\mathbf{x})\|_{L^2(K)}.\end{aligned}$$

The first term is estimated as previously. The second term is estimated as follows

$$\begin{aligned}\|\varphi_z(\oint_{\omega_z\cap\Omega_{\ell(K)}} v_i - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i\, d\mathbf{x})\|_{L^2(K)} &\leq \|\varphi_z\|_{L^2(K)}|\oint_{\omega_z\cap\Omega_{\ell(K)}} v_i - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i\, d\mathbf{x})| \\ &\leq ch_K|\oint_{\omega_z\cap\Omega_{\ell(K)}} v_i - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i\, d\mathbf{x})|.\end{aligned}$$

Consider first the case where the two subdomains $\Omega_{\ell(z)}$ and $\Omega_{\ell(K)}$ are adjacent, i.e. they share a common edge that we denote by e

- As a subcase, we still have $\ell(K) \neq \ell(z)$ and we consider z such that $z \notin \overline{\gamma}$, $\gamma \in \cup_{i=1}^{I}\gamma_i$ being a crack between the two subdomains $\Omega_{\ell(z)}$ and $\Omega_{\ell(K)}$. Then it is an endpoint of e which is the entire edge of two elements, each in one subdomain. Using the regularity of the triangulations $\mathcal{T}_h^\ell$, $\ell = \ell(K), \ell(z)$, we have

$$\begin{aligned}h_K|\oint_{\omega_z\cap\Omega_{\ell(K)}} v_i - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i\, d\mathbf{x})| &\leq c\, h_e^{\frac{1}{2}}\|\oint_{\omega_z\cap\Omega_{\ell(K)}} v_i - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i\, d\mathbf{x})\|_{L^2(e)} \\ &\leq c\, h_e^{\frac{1}{2}}(\|\oint_{\omega_z\cap\Omega_{\ell(K)}} v_i - v_i\|_{L^2(e)} + \|v_i - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i\|_{L^2(e)}).\end{aligned}$$

Denoting by κ any of the two elements K, K' which share the edge e, and $\ell(\kappa) = \ell(z)$ or $\ell(K)$, and thanks to the trace theorem [[48], Lemma 3.2] we have

$$\|\varphi\|_{L^2(e)} \leq c(h_e^{-\frac{1}{2}}\|\varphi\|_{L^2(\kappa)} + h_e^{\frac{1}{2}}|\varphi|_{H^1(\kappa)}), \tag{73}$$

and we obtain

$$\begin{aligned}h_e^{\frac{1}{2}}\|v_i - \oint_{\omega_z\cap\Omega_{\ell(\kappa)}} v_i\|_{L^2(e)} &\leq c\left(\|v_i - \oint_{\omega_z\cap\Omega_{\ell(\kappa)}} v_i\, d\mathbf{x}\|_{L^2(\kappa)} + h_e|v_i|_{H^1(\kappa)}\right) \\ &\leq c\, h_K\, \alpha_K^{-\frac{1}{2}}\|v_i\|_{1,\alpha,\Delta_K}.\end{aligned}$$

- If $z \in \overline{\gamma}$ and $z \in \mathcal{N}_C^- \setminus \mathcal{N}_C^+$, it follows from the Assumption 1, that z is an endpoint of e which is an entire edge of K and only a part of $\tilde{e}$ an edge of $\tilde{K}$. In addition $\mathbf{v} = (v_i)$, $i = 1, 2$ is not continuous through γ, therefore, we have to modify the previous argument. Inserting and subtracting $h_e^{-1}\int_e v_{i|\Omega_{\ell(\kappa)}}$,

$\kappa = (K, \tilde{K})$, we obtain

$$\begin{aligned} h_K \Big| \oint_{\omega_z\cap\Omega_{\ell(K)}} v_i - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i \, d\mathbf{x} \Big| \leq\ & h_K \big(\Big| \oint_{\omega_z\cap\Omega_{\ell(K)}} v_i - h_e^{-1} \int_e v_{i|\Omega_{\ell(K)}} \, d\tau \Big| \\ & + \Big| h_e^{-1} \int_e v_{i|\Omega_{\ell(z)}} \, d\tau - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i \Big| + h_e^{-1} \Big| \int_e [v_i] \ d\tau \Big| \big), \\ \leq\ & h_K \big(h_e^{-1} \int_e (|v_{i|\Omega_{\ell(K)}} - \oint_{\omega_z\cap\Omega_{\ell(K)}} v_i \, d\mathbf{x}|) \, d\tau \\ & + h_e^{-1} \int_e (|v_{i|\Omega_{\ell(z)}} - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i \, d\mathbf{x}|) \, d\tau + h_e^{-1} \Big| \int_e [v_i] \ d\tau \Big| \big), \\ \leq\ & c \, h_K \, \big(h_e^{-\frac{1}{2}} \| v_{i|\Omega_{\ell(K)}} - \oint_{\omega_z\cap\Omega_{\ell(K)}} v_i \, d\mathbf{x} \|_{L^2(e)} \\ & + h_e^{-\frac{1}{2}} \| v_{i|\Omega_{\ell(z)}} - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i \, d\mathbf{x} \|_{L^2(e)} + h_e^{-\frac{1}{2}} \| \, [v_i] \, \|_{L^2(e)} \big). \end{aligned}$$

Invoking the trace theorem (73) once, we obtain

$$\begin{aligned} h_K \Big| \oint_{\omega_z\cap\Omega_{\ell(K)}} v_i - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i \, d\mathbf{x} \Big| \leq\ & c \, h_K \, \big((h_e^{-1} \| v_i - \oint_{\omega_z\cap\Omega_{\ell(K)}} v_i \, d\mathbf{x} \|_{L^2(\kappa)} + |v_i|_{H^1(\kappa)}) \\ & + (h_e^{-1} \| v_i - \oint_{\omega_z\cap\Omega_{\ell(z)}} v_i \, d\mathbf{x} \|_{L^2(\kappa)} + |v_i|_{H^1(\kappa)}) \\ & + h_e^{-\frac{1}{2}} \| \, [v_i] \, \|_{L^2(e)} \big). \end{aligned}$$

The two first terms yield as previously

$$h_e^{-1} \| v_i - \oint_{\omega_z\cap\Omega_{\ell(\kappa)}} v_i \, d\mathbf{x} \|_{L^2(\kappa)} + |v_i|_{H^1(\kappa)}) \leq c \, h_K \, \alpha_\kappa^{-\frac{1}{2}} \| v_i \|_{1,\alpha,\Delta_\kappa}.$$

Thanks to assumption 1, the last term is bounded as follows

$$\begin{aligned} h_K \, (h_e^{-\frac{1}{2}} \, \| \, [v_i] \, \|_{L^2(e)}) \leq\ & h_K \, (h_e^{-\frac{1}{2}} \, \| v_{i|K} - R_h v_i \|_{L^2(e)} + (\frac{h_e}{\tilde{h}_e})^{\frac{1}{2}} \, (\tilde{h}_e^{-\frac{1}{2}} \, \| v_{i|\tilde{K}} - R_h v_i \|_{L^2(\tilde{e})})) \\ \leq\ & c \, (h_e^{\frac{1}{2}} \, \| v_{i|K} - R_h v_i \|_{L^2(e)} + \tilde{h}_e^{\frac{1}{2}} \| v_{i|\tilde{K}} - R_h v_i \|_{L^2(\tilde{e})}) \end{aligned}$$

The second term on the right hand side of this inequality is bounded for small $\delta > 0$ as follows

$$\tilde{h}_e^{\frac{1}{2}} \| v_{i|\tilde{K}} - R_h v_i \|_{L^2(\tilde{e})} \leq \delta \, \| v_{i|\tilde{K}} - R_h v_i \|^2_{L^2(\tilde{K})} + \frac{1}{4\delta} h_{\tilde{e}}, \tag{74}$$

and we have obtained for $\tilde{K}$

$$\| v_{i|\tilde{K}} - R_h v_i \|_{L^2(\tilde{K})} \leq c \, h_K \alpha_K^{-\frac{1}{2}} \| v_i \|_{1,\alpha,\Delta_{\tilde{K}}}.$$

Therefore the worst term is $O(h_K)$. The first term is simply added to $\|v_i - R_h v_i\|_{L^2(K)}$ on the left hand side of the estimate thanks to the decomposition (74) which will still add to the right hand side a term of $O(h_K)$.

- If $z \in \mathcal{N}_C^+$, z is an endpoint of e which is an entire edge of K and only a part of $\tilde{e}$ an edge of $\tilde{K}$. We denote by $\mathbf{a}_j$, $j = 1, 2, 3$ the vertices of $\tilde{K}$, and by λ_i, the associated barycentric functions,

$$\begin{aligned}\|v_{i|\Omega_{\ell(K)}} - R_h v_i\|_{L^2(K)} &\le \sum_{j=1}^{3} |(v_i - R_h v_i)(\mathbf{a}_j)| \|\lambda_j\|_{L^2(\tilde{K})} \\ &\le c\, h_{\tilde{K}} \sum_{j=1}^{3} |(v_{i|\Omega_{\ell(K)}} - R_h v_i)(\mathbf{a}_j)|,\end{aligned}$$

Using Assumption 1, and the regularity of the triangulations, we obtain

$$\|v_{i|\Omega_{\ell(K)}} - \mathcal{R}_h v_i\|_{L^2(K)} \le h_{\tilde{e}}^{\frac{1}{2}} \|v_{i|\Omega_{\ell(z)}} - R_h v_i\|_{L^2(\tilde{e})} + h_e^{\frac{1}{2}} \| [v_i] \|_{L^2(e)}.$$

When the subdomains $\Omega_{\ell(K)}$ and $\Omega_{\ell(z)}$ are not adjacent, by using Assumption 2, we introduce the subdomains $(\Omega_\ell)_\ell$ which are on the path between $\Omega_{\ell(K)}$ and $\Omega_{\ell(z)}$. We apply the previous argument each pair of adjacent subdomains. This establishes the first estimate of the Lemma. The second one is proven in exactly the same way by noting that

$$\|\varphi_z\|_{L^2(e)} \le c h_e^{\frac{1}{2}}.$$

References

[1] M. Ainsworth, J.T.Oden, C. Lee—Local a posteriori error estimators for variational inequalities, Numer. Math. PDE **9**, (1993), 23-33.

[2] J.R.Barber—Elasticity. Second edition. Solid Mechanics and its Applications, **107**, Kluwer Academic Publishers Group, Dordrecht, 2002.

[3] Z. Belhachmi—A posteriori error estimates for the 3D stabilized mortar finite element method applied to the Laplace equation, Math. Model. Numer. Anal, **37**, **6**, (2003), 991-1013.

[4] Z. Belhachmi—Residual a posteriori error estimates for a 3D mortar finite element method: the Stokes system, IMA J. Numer. Anal. **24** (2004), **3**, 521-547.

[5] Z. Belhachmi, F. Ben Belgacem—Quadratic finite element for Signorini problem, Math. Comp. **72**, (2003), 83-104.

[6] Z. Belhachmi, J.M. Sac-Epée, J. Sokolowski— Mixed finite element methods for a smooth domain formulation of a crack problem. SIAM J. Numer. Anal., **43**, **3** (2005), 1295–1320.

[7] F. Ben Belgacem—Numerical simulation of some variational inequalities arisen from unilateral contact problems by finite element method, Siam J. Numer. Anal, **37**, (2000), 1198-1216.

[8] F. Ben Belgacem, P. Hild, P. Laborde—Extention of the mortar finite element method to a variational inequality modeling unilateral contact, Math. Models Methods. Appl. Sci., **9**, (1999), 287-303.

[9] F. Ben Belgacem, Y. Renard—Hybrid finite element methods for the Signorini problem, Math. Comput. **72**, (2003), 1117-1145.

[10] C. Bernardi, V. Girault—A local regularization operator for triangular and quadrilateral finite elements, SIAM. J. Numer. Anal., **35** (1998), 1893–1916.

[11] C. Bernardi, Y. Maday, A.T. Patera—A new nonconforming approach to domain decomposition: the mortar element method, Collège de France Seminar , H. Brézis, J.L. Lions, Pitman, (1994), 13-51.

[12] C. Bernardi, R. Verfürth—adaptive finite element methods for elliptic equations with non-smooth coefficients, Numer. Math. **85** (2000), 579-608.

[13] D. Bucur, G. Buttazzo—Variational methods in shape optimization problems,

[14] F. Brezzi, W. W. Hager and P. A. Raviart— Error estimates for the finite element solution of variational inequalities, Numer. Math., **28**, (1977), 431-443.

[15] F. Brezzi, W. W. Hager and P. A. Raviart— Error estimates for the finite element solution of variational inequalities, Part 2: Mixed methods, Numer. Math., **31**, (1978), 1-16.

[16] C. Carstensen, O. Scherf, P. Wriggers—Adaptive finite elements for elastic bodies in contact; SIAM J. Sci. Comput. **20** (1999), 1605-1629.

[17] Z. Chen, R.H. Nochetto—Residual type a posteriori error estimates for elliptic obstacle problems, Numer. Math. **84**, (2000), 527-548.

[18] P.G. Ciarlet— Basic Error Estimates for Elliptic Problems, in the Handbook of Numerical Analysis, Vol **II**, P.G. Ciarlet & J.-L. Lions eds, North-Holland, (1991), 17-351.
[19] P. Clément—Approximation by finite element functions using local regularization, RAIRO Anal. Numér. **9** (1975), 77-84.
[20] P. Coorevits, P. Hild, K. Lhalouani, T. Sassi—Mixed finite element methods for unilateral problems: convergence analysis and numerical studies. Math. Comp., 71, 237, (2001), 1-25.
[21] P. Coorevits, P. Hild, J.-P. Pelle—A posteriori error estimators for unilateral contact with matching and nonmatching meshes, Comput. Methods Appl. Mech. Engrg. **186** (2000), 65-83.
[22] M. Crouzeix, V. Thomée—Resolvent estimates in l_p for discrete Laplacians on irregular meshes and maximum-norm stability of parabolic finite difference schemes. Comput. Methods Appl. Math. **1** (2001), **1**, 3-17.
[23] G. Duvaut and J.-L. Lions— *Les inéquations en mécanique et en physique*, Dunod, (1972).
[24] R.H.W. Hoppe, R. Kornhuber—Adaptive multilevel methods for obstacle problems, Siam. J. Numer. Anal. **31** (1994), 301-323.
[25] R. S. Falk—Error estimates for the approximation of a class of variational inequalities, Math. of Comp., **28** (1974), 963–971.
[26] V. Girault, P.-A. Raviart—Finite element methods for the Navier-Stokes equations, Theory and algorithms. Springer-Verlag (1986).
[27] R. Glowinski—Lectures on numerical methods for nonlinear variational problems, Springer, Berlin, (1980).
[28] P. Grisvard—Elliptic Problems in Nonsmooth Domains, Monographs and Studies in Mathematics, **24**, Pitman, 1985.
[29] J. Haslinger, I. Hlaváček, J. Nečas—Numerical Methods for Unilateral Problems in Solid Mechanics, in the Handbook of Numerical Analysis, Vol. **IV**, Part **2**, P.G. Ciarlet & J.-L. Lions eds, North-Holland, (1996).
[30] J. Haslinger, R. A. E. Mkinen—Introduction to shape optimization. Theory, approximation, and computation. Advances in Design and Control, textbf7, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003.
[31] F. Hecht, O. Pironneau—FreeFem++, see www.freefem.org.
[32] P. Hild, S. Nicaise—A posteriori error estimations of residual type for Signorini problems, Numer. Math. **101** (2005), **3**, 523-549.
[33] P. Hild, S. Nicaise—Residual a posteriori error estimators for contact problems in elasticity, M2AN **41** (2007), **5**, 897-923.
[34] M. Hintermüller, K. Ito, K. Kunisch—The dual-primal active set strategy as a semi-smooth Newton method, Siam J. Optim. **13**, **3** (2003), 865-888.
[35] S. Hüeber, B. I. Wohlmuth—An optimal error estimate for nonlinear contact problems, Siam J. Numer. Anal., **43** (2005), 156-176.
[36] K. Ito, K, Kunisch—Semi-smooth Newton methods for variatonal inequalities of the first kind, M2AN, Vol. **37**, No **1**, (2003), pp. 41?62
[37] C. Johnson—Adaptive finite element methods for obstacle problems, Math. Models Methods Appl. Sci. **2** (1992), 483-487.
[38] A.M. Khludnev, V.A. Kovtunenko— Analysis of cracks in solids, Southampton-Boston, WIT press, (2000).
[39] N. Kikuchi, J. Oden—- Contact problems in elasticity: A study of variational inequalities and finite element methods, SIAM, 1988.
[40] D. Kinderlehrer, G. Stamppachia—An introduction to variational inequalities and their applications, Academic Press, (1980).
[41] K. Lhalouani, T. Sassi—Nonconforming mixed variational formulation and domain decomposition for unilateral problems, East-West J. Numer. Math., 7, (1999), 23-30.
[42] J.-L. Lions, E. Magenes—Problèmes aux limites non homogènes, Dunod, (1968).
[43] M. Moussaoui, K. Khodja—Régularité des solutions d'un problème mêlé Dirichlet-Signorini dans un domaine polygonal plan, Commun. Part. Diff. Eq, **17**, (1992), 805-826.
[44] L. Slimane, A. Bendali, P. Laborde—Mixed formulations for a class of variational inequalities, M2AN, **38**, 1, (2004), 177-201.
[45] M. Sofonea, W. Han, M. Shillor—Analysis and approximation of contact problems with adhesion or damage. Pure and Applied Mathematics, **276**. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[46] S. Tahir—Problèmes de contact unilatéral et maillages incompatibles, PhD Thesis Université Paul Verlaine, Metz (2006)

[47] S. Tahir, Z. Belhachmi—Mixed finite elements discretizations of some variational inequalities arising in elasticity problems in domains with cracks, (2004-Fez conference on Differential Equations and Mechanics). Electron. J. Diff. Eqns., Conference **11**, (2004), 33-40.

[48] R. Verfürth—A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiely & Teubner (1996).

[49] B. Wohlmuth—An a posteriori error estimator for two-body contact problems on non-matching meshes, J. Sci. Comput. **33** (2007), **1**, 25–45.

[50] Z.-H. Zhong. *Finite Element Procedures for Contact-Impact Problems*, Oxford University Press, 1993.

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